Thursday, October 25, 2012
11:05 AM

Lecture 15

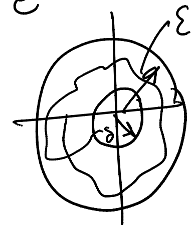
Stability of hybrid and switched systems

- Define notions of Stability
- Sufficient conditions

Def 1 Lyapunov Stability

$\forall \varepsilon > 0 \ \exists \delta_1 = \delta_1(\varepsilon)$ st $\forall \alpha \in Exec_{\mathcal{A}}$

$|\alpha(0)| \leq \delta_1 \Rightarrow \forall t \in \alpha.\text{ltime} \ \ |\alpha(t)| \leq \varepsilon$



Remark 1 if $\mathcal{A}$ Lyapunov Stable then for any $\varepsilon > 0$ if we choose $\Theta_A \subseteq B_{\delta_1(\varepsilon)}$ B_ε is an invariant

$x \in \mathbb{R}^n$

$r \in \mathbb{R}_{\geq 0}$

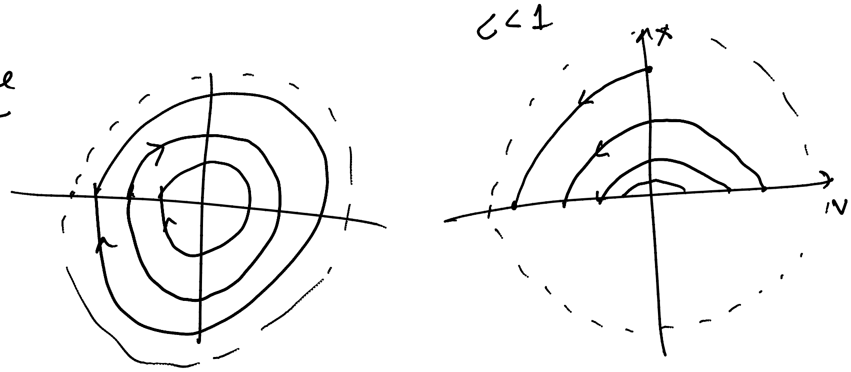
$B_{\bar{x}, r} = \{x \in \mathbb{R}^n \mid |\bar{x} - x| \leq r\}$

$\bar{x} = 0$

$B_{, r}$

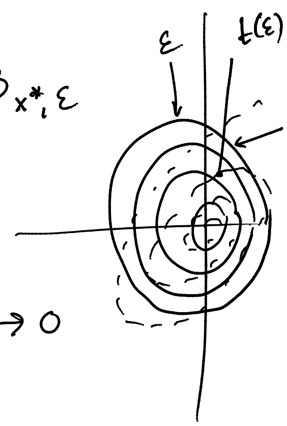
Example

$c < 1$



Convergence $\alpha(t) \to x^*$, $x^* \in \mathbb{R}^n$ as $t \to \infty$

$\forall \varepsilon > 0 \ \exists t = t(\varepsilon) \ \forall t' \geq t \ \ \alpha(t') \subseteq B_{x^*, \varepsilon}$



Def 2 (Asymptotic Stability)

(1) Lypunov Stable and

(2) $\exists \delta_2 > 0$ s.t $|\alpha(0)| \leq \delta_2 \Rightarrow t \to \infty \ |\alpha(t)| \to 0$

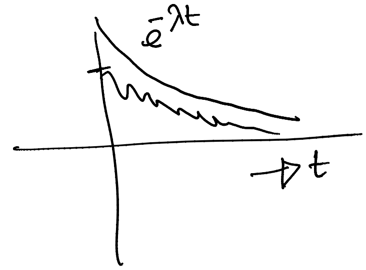
Globally AS if this hold for all δ_2

Def 3 (Exponential Stability)

$\exists c, c, \lambda > 0$ $\quad |\alpha(t)| \leq c e^{-\lambda t} |\alpha(0)|$

such that $|\alpha(0)| \leq \delta \Rightarrow \forall t \quad |\alpha(t)| \leq \ldots$

Globally ES if this holds for all δ.



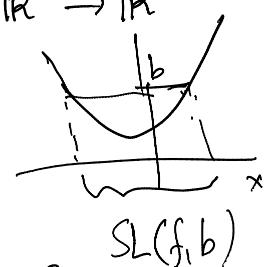
Sufficient Conditions for proving Stability

Lyapunov functions

$\dot{x} = f(x)$ (1) dynamical system $x \in \mathbb{R}^n$ $f : \mathbb{R}^n \to \mathbb{R}^n$

Solutions of (1) $\gamma : [0,T] \to \mathbb{R}^n$ $\gamma(t)$

$\dot{x} = Ax$ $\gamma(t).x = e^{At}\, \gamma(0).x$



C' : class of Continuously differentiable functions

f is positive definite if $f(x) > 0 \;\; \forall x \neq 0$

f is radially Unbounded if $f(x) \to \infty$ as $x \to \infty$

Level sets of $f : \mathbb{R}^n \to \mathbb{R}$ $L(f,b) \triangleq \{x \in \mathbb{R}^n \mid f(x) = b\}$

$SL(f,b) \triangleq \{x \in \mathbb{R}^n \mid f(x) \leq b\}$

Def Lyapunov fn. $v : \mathbb{R}^n \to \mathbb{R}$ $v \in C'$

(1) v is positive definite

(2) $\dot{v} < 0 \equiv \dfrac{\partial v}{\partial x} f(x) < 0 \quad x \neq 0$

Weak LF (2') $\dot{v} \leq 0 \quad \forall x \in \mathbb{R}^n$

$v : \mathbb{R}^n \to \mathbb{R}$

$\dot{v}$?

$\dot{v}(\gamma(t)) = \dfrac{\partial v}{\partial x} \dfrac{dx}{dt}$

$\dot{x} = f(x)$ $= \dfrac{\partial v}{\partial x} f(x) \leq 0 \quad \forall x$

Example: $v : \mathbb{R} \to \mathbb{R}$

$v(x) \triangleq x^2$

$\dot{v}$

$\gamma.x(t) = t^2 + 2t$

$\ldots = (t^2 + 2t)^2$

$\tau . x(t) = t^2 + 2t$

$V(x(t)) = (t^2 + 2t)^2$

$\dot{V}(x(t)) = \frac{d}{dt}(t^2+2t)^2 = \frac{\partial}{\partial x}(x^2) \cdot f(x)$

Not < 0

Lemma 1 $\forall V$ weak Lyapunov fn. of $\dot{x} = f(x)$ and any $b \geq 0$ $SL(V,b)$ is an invariant (if start state is in $SL(V,b)$)

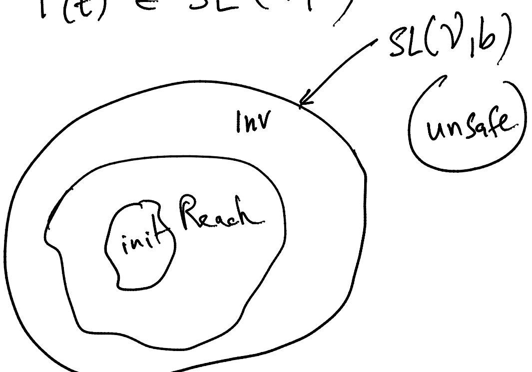
Proof. Suppose $\tau : [0,T] \to \mathbb{R}^n$ is a solution of $\dot{x} = f(x)$.
$\tau(0) \in SL(V,b)$ we have to show that

$\forall t \leq T \quad \tau(t) \in SL(V,b)$

$V(\tau(t)) \leq V(\tau(0)) \leq b$ as $\dot{V} \leq 0$

$\Rightarrow \tau(t) \in SL(V,b)$ ☒

Remark



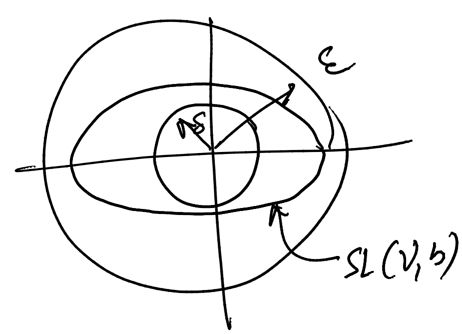
Thm 1 (i) if V is a WLF for $\dot{x} = f(x)$ then it is Lyapunov Stable
(ii) if V is a LF for $\dot{x} = f(x)$ then it is AS.

Proof 1(i) $\forall \varepsilon \; \exists \delta \; \ldots$

Fix ε.

Pick b s.t. $SL(V,b) \subseteq B_\varepsilon$

Then pick δ s.t. $B_\delta \subseteq SL(V,b)$



From Lemma 1 we know that as $\Theta \subseteq B_\delta$

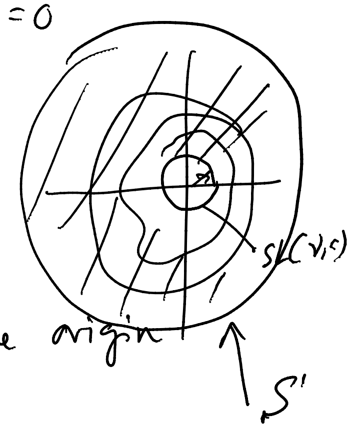
$SL(V,b)$ is an invariant $\Upsilon(t) \in B_\varepsilon$

Proof (ii) Υ is the solution of $\dot{x} = f(x)$

Since V is decreasing and its positive definite

as $t \to \infty$ $V(\Upsilon(t))$ must be converging to some limit, say c.

if $c = 0$ we are done $V(\Upsilon(t)) = 0 \Rightarrow \Upsilon(t) = 0$



we show that c cannot be > 0.

Suppose $c > 0$. $\dot{V}$

$\Upsilon(t)$ evolves outside $SL(V,c)$

in some compact set that does not contain the origin

$S = \{x \mid r \le x \le \varepsilon\}$ for some small r

$d \triangleq \max_{x \in S} \dot{V}(x)$ well defined because S is compact

$V(\Upsilon(t)) \le V(\Upsilon(0)) + dt$

and t is large enough then $V(\Upsilon(t)) < c$. (Contradicts) $\boxed{}$

Example Linear $\dot{x} = Ax$ $x \in \mathbb{R}^n$ $A \in \mathbb{R}^{n \times n}$

Fix Positive definite matrix $Q \in \mathbb{R}^{n \times n}$

$\Rightarrow A^T P + PA = -Q$ Lyapunov Equation

$P \in \mathbb{R}^{n \times n}$

$V(x) = x^T P x$

$\dot{V}(x) = -x^T Q x$

... A is Hurwitz

PD Matrix $M \in \mathbb{R}^{n \times n}$

M iff $\forall z \in \mathbb{R}^n$ $z \neq 0$

$z^T M z > 0$

i.e. all eigenvalues of A are in the left half