ECE 598: Modeling and Verifying Embedded Computing Systems Spring 2010

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Stability of Hybrid Systems

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21 Verifying Stability Properties

In the next wto lectures we study stability and stability verificaton of HIOAs. Informally, an HIOA is said to be stable if it converges to an equilibrium state starting from *any* state. Describing stability requires infinite (in fact uncountable) number of atomic propositions in temporal logics. Stability of each individual state model does not necessarily imply the stability of the whole automaton. The Lyapunov-based techniques we discuss rely on results from the literature on switched systems [HM99, Lib03, Bra98]. In the switched system model, details of the discrete mechanisms, namely, the preconditions and the effects of transitions, are neglected. Instead, an exogenous *switching signal* brings about the switches between the different state models. Assuming that the individual state models of a hybrid system we characterize the class of switching signals, based only on the rate of switches and not the particular sequence of switches, that guarantee stability of the whole system.

21.1 Assumptions

(1) Input/output variables and input actions are absent, that is, $U = Y = I = \emptyset$.

(2) The collection of locations/trajectory definitions is finite; the state models are indexed by a finite index set $I = \{1, \ldots, m\}$, for some $m \in \mathbb{N}$. The individual trajectory definitions of $\mathcal{A}$ are $\mathcal{S}_i$, $i \in I$.

(3) Let the continuous state space X_c be $\mathbb{R}^n$. For each state model $\mathcal{S}_i, i \in I$ the collection of V-DAIs F_i is described by differential equations in the vector notation of the form $d(\mathbf{x}_c) = f_i(\mathbf{x}_c)$, where f_i is a well behaved (locally Lipschitz) function.

(4) The trajectories defined by individual trajdefs converge to some equilibrium point in the state-space, say the origin, without loss of generality. Formally, $f_i(0) = 0$ for each $i \in I$.

21.1.1 Stability Definitions

Stability is a property of the continuous variables of HIOA $\mathcal{A}$, with respect to the standard Euclidean norm in $\mathbb{R}^n$. The Euclidean norm of $\alpha(t)$, denoted by $|\alpha(t)|$, is restricted to the set of real-valued continuous variables. Recall that the shorthand notation $\alpha(t)$ denotes the valuation of the state variables of an HIOA $\mathcal{A}$ in the execution α at time $t \in [0, \alpha.ltime]$.

Definition 1. *The origin is a* stable *equilibrium point of a HIOA $\mathcal{A}$, in the sense of Lyapunov, if for every $\epsilon > 0$, there exists a $\delta_1 = \delta_1(\epsilon) > 0$, such that for every closed execution α of $\mathcal{A}$, $|\alpha(0)| \leq \delta_1$ implies that $|\alpha(t)| \leq \epsilon$ for all t, $0 \leq t \leq \alpha.ltime$. In this case, we say that $\mathcal{A}$ is* stable.

For stable $\mathcal{A}$, the state can be bounded in an arbitrarily small ball of radius ϵ, by starting the automaton from a state within a suitably chosen smaller ball of radius δ_1.

Definition 2. *An HIOA $\mathcal{A}$ is* asymptotically stable *if it is stable and there exists $\delta_2 > 0$ so that for any execution fragment $|\alpha(0)| \leq \delta_2$ implies that $\alpha(t) \to 0$ as $t \to \infty$. If the above condition holds for all δ_2 then $\mathcal{A}$ is* globally asymptotically stable.

Examples. Stability and asymptotic stability.

Definition 3. *An HIOA $\mathcal{A}$ is said to be* exponentially stable *if there exist positive constants δ, c, and λ such that all closed executions fragments with $|\alpha(0)| \leq \delta$ satisfy the inequality $|\alpha(t)| \leq c|\alpha(0)|e^{-\lambda t}$, for all t, $0 \leq t \leq \alpha.ltime$. If the above holds for all δ then $\mathcal{A}$ is said to be* globally exponentially stable.

In the above definitions, the constants are quantified prior to the executions, and hence, these notions of stability are *uniform over executions*. We will employ the term "uniform" in the more conventional sense to describe uniformity with respect to the initial time of observation. Thus, *uniform stability* guarantees that the stability property in question holds not just for all executions, but for all reachable execution fragments.

Definition 4. *An HIOA $\mathcal{A}$ is* uniformly stable *in the sense of Lyapunov, if for every $\epsilon > 0$ there exists a constant $\delta_1 = \delta_1(\epsilon) > 0$, such that for any reachable closed execution fragment α, $|\alpha(0)| \leq \delta_1$ implies that $|\alpha(t)| \leq \epsilon$, for all t, $0 \leq t \leq \alpha.ltime$.*

Definition 5. *An HIOA $\mathcal{A}$ is said to be uniformly asymptotically stable if it is uniformly stable and there exists $\delta_2 > 0$, such that for every $\epsilon > 0$ there exists a $T > 0$, such that for any reachable execution fragment α,*

$$|\alpha(0)| \leq \delta_2 \Rightarrow |\alpha(t)| \leq \epsilon, \ \ \forall t \geq T \tag{1}$$

It is said to be globally uniformly asymptotically stable (GUAS) *if the above holds for all δ_2, with $T = T(\delta_2, \epsilon)$.*

Definition 6. *An HIOA $\mathcal{A}$ is* uniformly exponentially stable *if it is uniformly stable and there exist δ, c, and λ, such that for any reachable closed execution fragment α, if $|\alpha(0)| \leq \delta$ then $|\alpha(t)| \leq c|\alpha(0)|e^{-\lambda t}$, for all $0 \leq t \leq \alpha.ltime$. $\mathcal{A}$ is* globally uniformly exponentially stable *if the above holds for all δ with constant c and λ.*

21.2 Multiple Lyapunov Functions

A continuously differentiable function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ is said to be *positive definite* if $\mathcal{V}(0) = 0$ and $\mathcal{V}(\mathbf{x}_c) > 0$ for all $\mathbf{x}_c \neq 0$. If $\mathcal{V}(\mathbf{x}_c) \to \infty$ as $|\mathbf{x}_c| \to \infty$ then $\mathcal{V}$ is said to be radially unbounded. For $i \in I$, $\mathcal{V}_i : \mathbb{R}^n \to \mathbb{R}$ is said to be a *Lyapunov function* for mode i if 1. it is positive definite, 2. $\dot{V}(\mathbf{x}_c) \triangleq \frac{\partial \mathcal{V}}{\partial t} f_i(\mathbf{x}_x) < 0$ for all $\mathbf{x}_c \neq 0$.

If there exists a Lyapunov function $\mathcal{V}_i$ for $i \in \mathcal{I}$ then mode i is asymptotically stable. Furthermore, if $\mathcal{V}_i$ is radially unbounded then i is globally asymptotically stable.

If there exists positive definite continuously differentiable function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ and a positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ such that for each $i \in I$ $\frac{\partial \mathcal{V}}{\partial t} f_i(\mathbf{x}_x) < -W(\mathbf{x}_c)$ for all $\mathbf{x}_c \neq 0$, then $\mathcal{V}$ is said to be a *common Lyapunov function* for $\mathcal{A}$

Theorem 1. *$\mathcal{A}$ is GUAS if there exists a common Lyapunov function.*

In the absence of a common lyapunov function the stability verification of $\mathcal{A}$ has to rely of the discrete transitions (mode switches). The following theorem gives such a stability in terms of *multiple Lyapunov function*.

Theorem 2. *Let $\mathcal{V}_i, i \in I$ be a collection of radially bounded Lyapunov functions for the i modes of $\mathcal{A}$. Suppose for any $i \in I$ there exist collection of positive definite functions $W_i : \mathbb{R}^n \to \mathbb{R}$ such that, for any execution α, and for any time t_1, t_2 such that $\alpha(t_1).\mathbf{x}_d = i$ $\alpha(t_1).\mathbf{x}_d = i$ and for all $t_1 < t < t_2$, $\alpha(t).\mathbf{x}_d \neq i$,*

$$\mathcal{V}_i(\alpha(t_2).\mathbf{x}_c) - \mathcal{V}_i(\alpha(t_1).\mathbf{x}_c) \leq -W_i(\alpha(t_1).\mathbf{x}_c).$$

Then, $\mathcal{A}$ is GUAS.

21.2.1 ADT Theorem of Heshpanha and Morse

The notion of *average dwell time (ADT)* [HM99] precisely defines a restricted classes of switching signals that guarantee stability of a switched system. A large average dwell time means that the system spends enough time in each mode, so as to dissipate the transient energy gained through mode switches. This itself is not sufficient for stability; in addition, the individual modes of the automaton must also be stable. Translated to HIOAs: given an HIOA $\mathcal{A}$ such that the individual state models of $\mathcal{A}$ are stable, if the ADT of $\mathcal{A}$ is greater than a certain constant (a function of the state model dynamics), then $\mathcal{A}$ is stable. However, application of this criterion relies on checking that the ADT of $\mathcal{A}$ is greater than some constant—a property that depends on the rate of mode switches over *all* executions of $\mathcal{A}$.

Definition 7. *Let $\mathcal{A}$ be an HIOA with state models indexed by a finite set I. A discrete transition $\mathbf{x} \xrightarrow{a} \mathbf{x}'$ of $\mathcal{A}$ is said to be a* mode switch *if for some $i, j \in I, i \neq j$, $\mathbf{x} \in Inv_i$ and $\mathbf{x}' \in Inv_j$. The set of mode switching transitions of $\mathcal{A}$ is denoted by $\mathcal{M}$. Given an execution fragment α of $\mathcal{A}$, the number of mode switches over α is denoted by $N(\alpha)$.*

A discrete transition is a mode switch if its pre- and post-states satisfy invariants of different different state models. This implies that different sets of differential equations guide the evolution of the continuous variables, before and after a mode switch.

Definition 8. *Given a duration of time $\tau_a > 0$, HIOA $\mathcal{A}$ has* Average Dwell Time (ADT) *τ_a if there exists a positive constant N_0, such that for every reachable execution fragment α,*

$$N(\alpha) \leq N_0 + \alpha.ltime/\tau_a, \tag{2}$$

The number of extra switches *of α with respect to τ_a is defined as $S_{\tau_a}(\alpha) := N(\alpha) - \alpha.ltime/\tau_a$.*

Lemma 3. *Suppose $\mathcal{A}$ is an HIOA and $\tau_a > 0$ is an average dwell time for $\mathcal{A}$. Then, any τ'_a that is $0 \le \tau'_a < \tau_a$ is also an average dwell time of $\mathcal{A}$*

Proof. Inequality (2) is satisfied if we replace τ_a with a smaller τ'_a. ∎

Theorem 1 from [HM99], adapted to HIOA, gives a sufficient condition for stability based on average dwell time. Informally, it states that a hybrid system is stable if the discrete switches are between modes which are individually stable, provided that the switches do not occur too frequently on the average.

Theorem 4. *Suppose there exist positive definite, continuously differentiable functions $\mathcal{V}_i : \mathbb{R}^n \to \mathbb{R}^n$, for each $i \in I$, such that we have two positive numbers λ_0 and μ, and two strictly increasing continuous functions β_1, β_2 such that:*

$$\beta_1(|\mathbf{x}_c|) \le \mathcal{V}_i(\mathbf{x}_c) \le \beta_2(|\mathbf{x}_c|), \quad \forall \mathbf{x}_c, \quad \forall i \in I, \tag{3}$$

$$\frac{\partial \mathcal{V}_i}{\partial \mathbf{x}_c} f_i(\mathbf{x}_c) \le -2\lambda_0 \mathcal{V}_i(\mathbf{x}_c), \quad \forall \mathbf{x}_c, \quad \forall i \in I, \textit{ and} \tag{4}$$

$$\mathcal{V}_i(\mathbf{x}'_c) \le \mu \mathcal{V}_j(\mathbf{x}_c), \quad \forall \mathbf{x} \xrightarrow{a}_{\mathcal{A}} \mathbf{x}', \textit{ where } i = \mathbf{x}' \lceil \textit{mode and } j = \mathbf{x} \lceil \textit{mode}. \tag{5}$$

Then, $\mathcal{A}$ is globally uniformly asymptotically stable if it has an ADT $\tau_a > \frac{\log \mu}{\lambda_0}$.

Its worth making a few remarks about this theorem. First of all, it is well-known that if the state model $\mathcal{S}_i, i \in I$ is globally asymptotically stable, then there exists a Lyapunov function $\mathcal{V}_i$ that satisfies (3) and $\frac{\partial \mathcal{V}_i}{\partial \mathbf{x}_c} f_i(\mathbf{x}_c) \le -2\lambda_i V_i(\mathbf{x}_c)$, for appropriately chosen $\lambda_i > 0$. As the index set I is finite a λ_0 independent of i can be chosen such that for all $i \in I$, Equation (4) holds. The third assumption, Equation 5, restricts the maximum increase in the value of the current Lyapunov functions over any mode switch, by a factor of μ.

In [HM99] and [Lib03] this theorem is presented for the switched system model which differs from the more general HIOA model in two ways: (a) In the switched system model, all variables are continuous except for the *mode* variable which determines the active state model. In HIOA, there are both discrete and continuous variables. (b) The (discrete) transitions of a switched system correspond to the switching signal changing the value of *mode*; values of continuous variables remain unchanged over transitions. In HIOAs, transitions can change the value of continuous variables. For example, a stopwatch is typically modeled as a continuous variable that is reset by discrete transitions. The proof of Theorem 4 still works for the HIOA model because for this analysis, it suffices to consider only those discrete transitions of HIOAs that are also mode switches. Assumption (2) guarantees that non-mode switching transitions do not change the value of the continuous variables. Secondly, resetting continuous variables change the value of the Lyapunov functions but hypothesis 5 guarantees that the change is bounded by a factor of μ.

Proof sketch for Theorem 4. This proof is adapted from the proof of Theorem 3.2 of [Lib03] which constructs an exponentially decaying bound on the Lyapunov functions of each mode along any execution. Suppose α is any execution of $\mathcal{A}$. Let $T = \alpha.ltime$ and $t_1, \ldots, t_N$ be instants of mode switches in α. We will find an upper-bound on the value of $\mathcal{V}_{\alpha(T)\lceil mode}(\alpha(T))$, where $\alpha(t) \lceil mode \triangleq i, i \in I$ if and only if $\alpha(t) \in Inv_i$. We define a function $W(t) \triangleq e^{2\lambda_0 t}\mathcal{V}_{\alpha(t)\lceil mode}(\alpha(t))$. Using the fact that W is non-increasing between mode switches and Equation 5 it can be shown that $W(t_{i+1}) \leq \mu W(t_i)$. Iterating this inequality $N(\alpha)$ times we get $W(T) \leq \mu^{N(\alpha)} W(0)$, that is

$$\begin{aligned} e^{2\lambda_0 T}\mathcal{V}_{\alpha(T)\lceil mode}(\alpha(T)) &\leq \mu^N(\alpha)\mathcal{V}_{\alpha(0)\lceil mode}(\alpha(0)), \\ \mathcal{V}_{\alpha(T)\lceil mode}(\alpha(T)) &\leq e^{-2\lambda_0 T + N(\alpha)\log\mu}\mathcal{V}_{\alpha(0)\lceil mode}(\alpha(0)) \end{aligned}$$

If α has average dwell time τ_a, then

$$\begin{aligned} \mathcal{V}_{\alpha(T)\lceil mode}(\alpha(T)) &\leq e^{-2\lambda_0 T + (N_0 + \frac{T}{\tau_a})\log\mu}\mathcal{V}_{\alpha(0)\lceil mode}(\alpha(0)) \\ &\leq e^{N_0\log\mu} e^{(\frac{\log\mu}{\tau_a} - 2\lambda_0)T}\mathcal{V}_{\alpha(0)\lceil mode}(\alpha(0)). \end{aligned}$$

Now, if $\tau_a > \frac{\log\mu}{2\lambda}$ then $\mathcal{V}_{\alpha(T)\lceil mode}(\alpha(T))$ converges to 0 as $T \to 0$. Then from (3) it follows that $\mathcal{A}$ is globally asymptotically stable.

21.3 ADT Equivalence

In order to check whether τ_a is an ADT for a given HIOA $\mathcal{A}$, it is often easier to check the same ADT property for another, more abstract, HIOA $\mathcal{B}$ that is "equivalent" to $\mathcal{A}$ with respect to switching behavior. This notion of equivalence is formalized as follows.

Definition 9. *Given HIOAs $\mathcal{A}$ and $\mathcal{B}$, if for all $\tau_a > 0$, τ_a is an ADT for $\mathcal{B}$ implies that τ_a is an ADT for $\mathcal{A}$, then we say that $\mathcal{A}$* switches slower *than $\mathcal{B}$ and write this as $\mathcal{A} \leq_{switch} \mathcal{B}$. If $\mathcal{B} \leq_{switch} \mathcal{A}$ and $\mathcal{A} \leq_{switch} \mathcal{B}$ then we say $\mathcal{A}$ and $\mathcal{B}$ are* ADT-equivalent.

We propose an inductive method for proving ADT-equivalence. The key idea is to use a new variety of forward simulation relation that we encountered in Section **??**, in the context of verification of trace inclusions. Here, instead of the trace of an execution, we are concerned with the number of mode switches that occur and the amount of time that elapses over an execution.

Definition 10. *Consider HIOAs $\mathcal{A}$ and $\mathcal{B}$. A* switching simulation relation *from $\mathcal{A}$ to $\mathcal{B}$ is a relation $\mathcal{R} \subseteq Q_{\mathcal{A}} \times Q_{\mathcal{B}}$ satisfying the following conditions, for all states $\mathbf{x}$ and $\mathbf{y}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively:*

1. (Start condition) *If $\mathbf{x} \in \Theta_{\mathcal{A}}$ then there exists a state $\mathbf{y} \in \Theta_{\mathcal{B}}$ such that $\mathbf{x}\ \mathcal{R}\ \mathbf{y}$.*
2. (Transition condition) *If $\mathbf{x}\ \mathcal{R}\ \mathbf{y}$ and α is an execution fragment of $\mathcal{A}$ with $\alpha.fstate = \mathbf{x}$ and consisting of one single action surrounded by two point trajectories, then $\mathcal{B}$ has a closed execution fragment β, such that $\beta.fstate = \mathbf{y}$, $N(\beta) \geq 1$, $\beta.ltime = 0$, and $\alpha.lstate\ \mathcal{R}\ \beta.lstate$.*
3. (Trajectory condition) *If $\mathbf{x}\ \mathcal{R}\ \mathbf{y}$ and α is an execution fragment of $\mathcal{A}$ with $\alpha.fstate = \mathbf{x}$ and consisting of a single closed trajectory τ of a particular state model $\mathcal{S}$, then $\mathcal{B}$ has a closed execution fragment β, such that $\beta.fstate = \mathbf{y}$, $\beta.ltime \leq \alpha.ltime$, and $\alpha.lstate\ \mathcal{R}\ \beta.lstate$.*

Note that HIOAs $\mathcal{A}$ and $\mathcal{B}$ are not necessarily comparable.

Lemma 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be HIOAs, and let $\mathcal{R}$ be a switching simulation relation from $\mathcal{A}$ to $\mathcal{B}$, then for all $\tau_a > 0$ and for every execution α of $\mathcal{A}$, there exists an execution β of $\mathcal{B}$ such that $S_{\tau_a}(\alpha) \leq S_{\tau_a}(\beta)$.*

Proof. We fix τ_a and α and construct an execution of $\mathcal{B}$ that has more extra switches than α. Let $\alpha = \tau_0 a_1 \tau_1 a_2 \tau_2 \ldots$ and let $\alpha.fstate = \mathbf{x}$. We consider cases:

Case 1: α is an infinite sequence. We can write α as an infinite concatenation $\alpha_0 \frown \alpha_1 \frown \alpha_2 \ldots$, in which the execution fragments α_i with i even consist of a trajectory only, and the execution fragments α_i with i odd consist of a single discrete transition surrounded by two point trajectories.

We define inductively a sequence $\beta_0\beta_1\beta_2\ldots$ of closed execution fragments of $\mathcal{B}$ such that $\mathbf{x}\ \mathcal{R}\ \beta_0.fstate$, $\beta_0.fstate \in \Theta_{\mathcal{B}}$, and for all i, $\beta_i.lstate = \beta_{i+1}.fstate$, $\alpha_i.lstate\ \mathcal{R}\ \beta_i.lstate$, and $S_{\tau_a}(\beta) \geq S_{\tau_a}(\alpha)$. Property 1 of Definition 10 ensures that there exists such a $\beta_0.fstate$ because $\alpha_0.fstate \in \Theta_{\mathcal{A}}$. We use Property 3 of Definition 10 for the construction of the β_i's with i even. This gives us $\beta_i.ltime \leq \alpha_i.ltime$ for every even i. We use Property 2 of Definition 10 for the construction of the β_i's with i odd. This gives us $\beta_i.ltime = \alpha_i.ltime$ and $N(\beta_i) \geq N(\alpha_i)$ for every odd i. Let $\beta = \beta_0 \frown \beta_1 \frown \beta_2 \ldots$. Since $\beta_0.fstate \in \Theta_{\mathcal{B}}$, β is an execution for $\mathcal{B}$. Since $\beta.ltime \leq \alpha.ltime$ and $N(\beta) \geq N(\alpha)$, the required property follows.

Case 2: α is a finite sequence ending with a closed trajectory. Similar to first case.

Case 3: α is a finite sequence ending with an open trajectory. Similar to first case except that the final open trajectory τ of α is constructed using a concatenation of infinitely many closed trajectories of $\mathcal{A}$ such that $\tau = \tau_0 \frown \tau_1 \frown \ldots$. Then, working recursively, we construct a sequence $\beta_0\beta_1\ldots$ of closed execution fragments of $\mathcal{B}$ such that for each i, $\tau_i.lstate \mathcal{R} \beta_i.lstate$, $\beta_i.lstate = \beta_{i+1}.fstate$, and $\beta_i.ltime \leq \tau_i.ltime$. This construction uses induction on i, using Property 3 of Definition 10 in the induction step. Now, let $\beta = \beta_0 \frown \beta_1 \frown \ldots$. Clearly, β is an execution fragment of $\mathcal{B}$ and $\tau.fstate \mathcal{R} \beta.fstate$ and $\beta.ltime \leq \tau.ltime$.

∎

Theorem 6. *If $\mathcal{A}$ and $\mathcal{B}$ are HIOAs and $\mathcal{R}$ is a switching simulation relation from $\mathcal{A}$ to $\mathcal{B}$, then $\mathcal{A} \leq_{switch} \mathcal{B}$.*

Proof. We fix a τ_a. Given N_0 such that for every execution β of $\mathcal{B}$, $S_{\tau_a}(\beta) \leq N_0$, it suffices to show that for every execution α of $\mathcal{A}$, $S_{\tau_a}(\alpha) \leq N_0$. We fix α. From Lemma 5 we know that there exists a β such that $S_{\tau_a}(\beta) \geq S_{\tau_a}(\alpha)$, from which the result follows. ∎

Corollary 7. *Let $\mathcal{A}$ and $\mathcal{B}$ be HIOAs. Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ be a switching forward simulation relation from $\mathcal{A}$ to $\mathcal{B}$ and from $\mathcal{B}$ to $\mathcal{A}$, respectively. Then, $\mathcal{A}$ and $\mathcal{B}$ are ADT-equivalent.*

Switching simulation relations and Corollary 7 give us an inductive method for proving that *any* given pair of HIOA are equivalent with respect to switching speed, that is, average dwell time. The theorem prover strategies for proving forward simulations can be used to partially automate switching simulation proofs. An interesting related question is computation of the switching simulation relation $\mathcal{R}$ from the specifications of $\mathcal{A}$ and $\mathcal{B}$.

Linear Hysteresis Switch. Consider LinHSwitch shown in Figure 1. The monitoring signals are generated by linear differential equations: for each $i \in I$, $d(\mu_i) = c_i \mu_i$ if $mode = i$, otherwise $d(\mu_i) = 0$; $c_i, i \in I$, is a positive constant. The switching logic unit implements the scale independent hysteresis switching.

Observe that the switching behavior of LinHSwitch does not depend on the value of the μ_i's but only on the ratio of $\frac{\mu_i}{\mu_{min}}$, which is always within $[1, (1+h)]$. Specifically, when LinHSwitch is in mode i, all the ratios remain constant, except $\frac{\mu_i}{\mu_{min}}$. The ratio $\frac{\mu_i}{\mu_{min}}$ increases monotonically from 1 to either $(1+h)$ or to $(1+h)^2$, in time $\frac{1}{c_i} \ln(1+h)$ or $\frac{2}{c_i} \ln(1+h)$, respectively. Based on this observation, we will first show that there exists a automaton $\mathcal{B}$, such that LinHSwitch $\leq_{switch} \mathcal{B}$, using a switching simulation relation.

Example: Abstract switching automaton. We begin by constructing the abstract automaton $\mathcal{B}$. Consider a graph $G = (\mathcal{V}, \mathcal{E}, w, e_0)$, where:

1. $\mathcal{V} \subset \{1, (1+h)\}^m$, such that for any $v \in V$, all the m-components are not equal. We denote the i^{th} component of $v \in V$ by $v[i]$.
2. An edge $(u, v) \in \mathcal{E}$ if and only if, one of the following conditions hold:
   (a) There exists $j \in \{1, \ldots, m\}$, such that, $u[j] \neq v[j]$ and for all $i \in \{1, \ldots, m\}$, $i \neq j$, $u[i] = v[i]$. The cost of the edge $w(u, v) := \frac{1}{c_j} \ln(1+h)$ and we define $\zeta(u, v) := j$.
   (b) There exists $j \in \{1, \ldots, m\}$ such that $u[j] = 1, v[j] = (1+h)$ and for all $i \in \{1, \ldots, m\}$, $i \neq j$ implies $u[i] = (1+h)$ and $v[i] = 1$. The cost of the edge $w(u, v) := \frac{2}{c_j} \ln(1+h)$ and we define $\zeta(u, v) := j$. The i^{th} component of the source (destination) vertex of edge e is denoted by $e[1][i]$ ($e[2][i]$, respectively).
3. $e_0 \in \mathcal{E}$, such that $e_0[1][i_0] = (1+h)$ and for all $i \neq i_0$, $e_0[1][i] = 1$.

A typical execution $\alpha = \tau_0, a_1, \tau_1, a_2, \tau_2$ of LinHSwitch is as follows: τ_0 is a point trajectory that maps to the state $(mode = 1, [\mu_1 = (1+h)C_0, \mu_2 = C_0, \mu_3 = C_0])$, $a_1 = \mathsf{switch}(1,3)$, $\tau_1.dom = [0, \frac{1}{c_3} \ln(1+h)]$, $(\tau_1 \downarrow \mu_3)(t) = C_0 e^{c_3 t}$, $a_2 = \mathsf{switch}(3,2)$, $\tau_2.dom = [0, \frac{2}{c_2} \ln(1+h)]$, $(\tau_2 \downarrow \mu_2)(t) = C_0 e^{c_2 t}$. Note that each edge e of G corresponds to a mode of LinHSwitch; this correspondence is captured by the ζ function in the definition of G.
We define a relation $\mathcal{R}$ on the states on $\mathcal{A} =$ LinHSwitch and $\mathcal{B} = \mathtt{Aut}(G)$.

Definition 11. *For any* $\mathbf{x} \in Q_{\mathcal{A}}$ *and* $\mathbf{y} \in Q_{\mathcal{B}}$, $\mathbf{x}\, \mathcal{R}\, \mathbf{y}$ *if and only if:*

1. $\zeta(\mathbf{y} \lceil mode) = \mathbf{x} \lceil mode$
2. *For all* $j \in \{1, \ldots, n\}$,
   (a) $\frac{\mathbf{x} \lceil \mu_j}{\mathbf{x} \lceil \mu_{min}} = e^{c_j (\mathbf{y} \lceil x)}$, *if* $j = \zeta(\mathbf{y} \lceil mode)$,
   (b) $\frac{\mathbf{x} \lceil \mu_j}{\mathbf{x} \lceil \mu_{min}} = (\mathbf{y} \lceil mode)[k][j]$, $k \in \{1, 2\}$.

Part 1 of Definition 11 states that if $\mathcal{A}$ is in $mode\ j$ and $\mathcal{B}$ is in mode e, then $\zeta(e) = j$. Part 2 states that for all $j \neq \zeta(e)$, the j^{th} component of $e[1]$ and $e[2]$ are the same, and are equal to μ_j / μ_{min}, and for $j = \zeta(e)$, $\mu_j = \mu_{min} e^{c_j x}$. Lemma 8 states that $\mathcal{R}$ is a switching simulation relation from $\mathcal{A}$ and $\mathcal{B}$. The proof follows the typical pattern of simulation proofs. We show by a case analysis that every action and state model of automaton $\mathcal{A}$ can be simulated by an execution fragment of $\mathcal{B}$ with at least as many extra switches. From Theorem 6 it follows that HIOA LinHSwitch $\leq_{switch}$ $\mathtt{Aut}(G)$, and therefore if τ_a is an ADT for $\mathtt{Aut}(G)$ then it is also an ADT for LinHSwitch.

Lemma 8. *$\mathcal{R}$ is a switching simulation relation from $\mathcal{A}$ to $\mathcal{B}$.*

21.4 Verifying ADT: Optimization-based Approach

We attempt to find an execution of the automaton that violates the ADT property. Failure to find such a counterexample execution indicates that the ADT property is satisfied by the HIOA. The search for a counterexample execution is formulated as an optimization problem. If we solve the

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automaton LinHSwitch($I$ : type, $i_0 : I, h$ : Real, $c$ : Array[$I$, Real])
where $h \geq 0$
  signature
    internal switch($i, j : I$) where $i \neq j$

  variables
    internal $mode : I := i_0; \mu$ : Array[$I$, Real];
        initially $\forall i : I, (i = i_0 \wedge \mu[i] = (1+h)C_0)$
          $\vee (i \neq i_0 \wedge \mu[i] = C_0)$
    let $\mu_{min} := \min_{i:I}\{\mu[i]\}$

  transitions
    internal switch($i, j$)
      pre $mode = i \wedge (1+h)\mu[j] \leq \mu[i]$;
      eff $mode := j$;

  trajectories
    trajdef mode($i : I$)
      invariant $mode = i$;
      stop when $\exists j : I, (1+h)\mu[j] \leq \mu[i]$;
      evolve $\forall j : I, (j = i \wedge d(\mu[j]) = c[j]\mu[j])$
          $\vee (j \neq i \wedge d(\mu[j]) = 0)$;
```

Figure 1: Linear hysteresis switch.

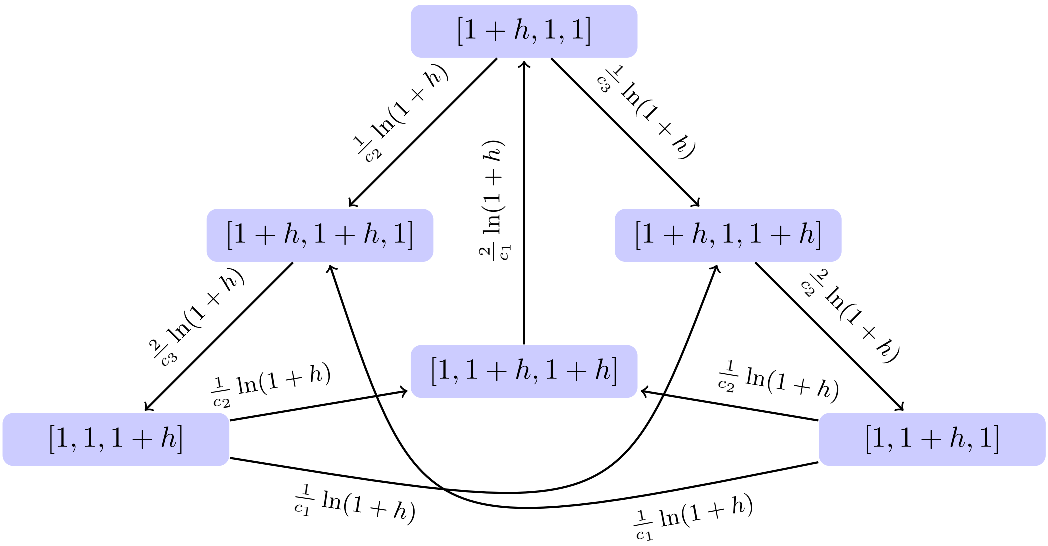

Figure 2: ADT-equivalent graph ($m = 3$) for LinHSwitch.

following optimization problem:

$$\mathsf{OPT}(\tau_a): \quad \alpha^* \in \arg \max_{\alpha \in \mathsf{Execs}_{\mathcal{A}}} S_{\tau_a}(\alpha),$$

and the optimal value $S_{\tau_a}(\alpha^*)$ turns out to be bounded, then we can conclude that $\mathcal{A}$ has ADT τ_a. Otherwise, if $S_{\tau_a}(\alpha^*)$ is unbounded then we can conclude that τ_a is not an ADT for $\mathcal{A}$. In fact, any execution α that gives an unbounded value of $\mathsf{OPT}(\tau_a)$ would serve as a counterexample execution violating the average dwell time property. We study particular classes of HIOA for which $\mathsf{OPT}(\tau_a)$ can be formulated and solved efficiently.

21.4.1 Initialized HIOA

An closed execution fragment α of an HIOA is said to be a *cyclic fragment* if $\alpha.fstate = \alpha.lstate$. The next theorem implies that for an initialized HIOA $\mathcal{A}$, it is necessary and sufficient to solve $\mathsf{OPT}(\tau_a)$ over the space of the cyclic fragments of $\mathcal{A}$ instead of the larger space of all execution fragments.

Theorem 9. *Given $\tau_a > 0$ and initialized HIOA $\mathcal{A}$, $\mathsf{OPT}(\tau_a)$ is bounded if and only if $\mathcal{A}$ does not have any cycles with extra switches with respect to τ_a.*

Proof. For simplicity we assume that all discrete transitions of the automaton $\mathcal{A}$ are mode switches and that for any pair of modes i, j, there exists at most one action which can bring about a mode switch from i to j. Existence of a reachable cycle α with extra switches with respect to τ_a is sufficient to show that τ_a is not an ADT for $\mathcal{A}$. This is because by concatenating a sequence of α's, we can construct an execution fragment $\alpha \frown \alpha \frown \alpha \ldots$ with an arbitrarily large number of extra switches.

We prove by contradiction that existence of a cycle with extra switches is necessary for making $\mathsf{OPT}(\tau_a)$ unbounded. We assume that $\mathsf{OPT}(\tau_a)$ is unbounded for $\mathcal{A}$ and that $\mathcal{A}$ does not have any cycles with extra switches. By the definition of OPT, for any constant N_0 there exists an execution that has more than N_0 extra switches with respect to τ_a. Let us choose $N_0 > |I|^3$. Of all the executions that have more than N_0 extra switches, let $\alpha = \tau_0 a_1 \tau_1 \ldots \tau_n$ be a closed execution that has the smallest number of mode switches. From α, we construct $\beta = \tau_0^* a_1 \tau_1^* \ldots \tau_n^*$, using the following two rules:

1. Each τ_i of α is replaced by: $\tau_i^* = \arg\min\{\tau.ltime \mid \tau.fstate \in R_{a_i}, \tau.lstate \in Pre_{a_{i+1}}\}$.
2. If there exists $i, j \in I$, such that $a_i = a_j$ and $a_{i+1} = a_{j+1}$, then we make $\tau_i^* = \tau_j^*$.

Claim 1. *The sequence β is an execution fragment of $\mathcal{A}$ and $S_{\tau_a}(\beta) > |I|^3$.*

Proof of claim: We prove the first part of the claim by showing that each application of the above rules to an execution fragment of $\mathcal{A}$ results in another execution fragment. Consider *Rule (1)* and fix i. Since $\tau_i^*.fstate \in R_{a_i}$ and $\tau_{i-1}.lstate \in Pre_{a_i}$, $\tau_{i-1}.lstate \overset{a_i}{\rightarrow} \tau_i^*.fstate$. And, since $\tau_i^*.lstate \in Pre_{a_{i+1}}$ and $\tau_{i+1}.fstate \in R_{a_{i+1}}$, we know that $\tau_i^*.lstate \overset{a_{i+1}}{\rightarrow} \tau_{i+1}.fstate$. Now for *Rule (2)*, we assume there exist i and j such that the hypothesis of the rule holds and suppose $\tau_j^* = \tau_i^* = \tau_i$. We know that even if $\tau_j^* \neq \tau_j$, the first states of both are in R_{a_j} and the last states are in $Pre_{a_{j+1}}$. Therefore, a_j matches up the states of τ_{j-1} and τ_j^* and likewise a_{j+1} matches the states of τ_j^* and τ_{j+1}.

The second part of the claim follows from the fact that each trajectory τ_i is replaced by the shortest trajectory τ_i^* from the initialization set of the previous transition R_{a_i} to the guard set of the next transition $Pre_{a_{i+1}}$. That is, for each i, $0 < i < n$, $\tau_i^*.ltime \leq \tau_i.ltime$ and therefore $\beta.ltime \leq \alpha.ltime$ and $S_{\tau_a}(\beta) > N_0 > |I|^3$.

Since $N(\beta) > |I|^3$, there must be a sequence of 3 consecutive modes that appear multiple times in β. That is, there exist $i, j \in \{1, \ldots, m\}$, and $p, q, r \in I$, such that $\tau_i^*.fstate \lceil mode = \tau_j^*.fstate \lceil mode = p$, $\tau_{i+1}^*.fstate \lceil mode = \tau_{j+1}^*.fstate \lceil mode = q$, and $\tau_{i+2}^*.fstate \lceil mode = \tau_{j+2}^*.fstate \lceil mode = r$. Then, from *Rule (2)* we know that $\tau_{i+1}^* = \tau_{j+1}^*$. In particular, $\tau_{i+1}^*.fstate = \tau_{j+1}^*.fstate$, that is, we can write $\beta = \beta_p \frown \gamma \frown \beta_s$, where γ is a cycle. Then we have the following:

$$N(\beta_p) + N(\gamma) + N(\beta_s) > N_0 + \beta_p.ltime/\tau_a + \gamma.ltime/\tau_a + \beta_s.ltime/\tau_a$$

$$N(\beta_p) + N(\beta_s) + S_{\tau_a}(\gamma) > N_0 + \beta_p.ltime/\tau_a + \beta_s.ltime/\tau_a$$

$$N(\beta_p \frown \beta_s) > N_0 + \beta_p \frown \beta_s.ltime/\tau_a \qquad [\beta_p.lstate = \beta_s.fstate]$$

The last step follows from the assumption that $S_{\tau_a}(\gamma) \leq 0$. Therefore, we have $S_{\tau_a}(\beta_p \frown \beta_s) > N_0$ which contradicts our assumption that β has the smallest number of mode switches among all the executions that have more than N_0 extra switches with respect to τ_a. ■

The following corollary allows us to limit the search for cycles with extra switches to cycles with at most $|I|^3$ mode switches. It is proved by showing that any cycle with extra switches that has more than $|I|^3$ mode switches can be decomposed into two smaller cycles, one of which must also have extra switches.

Corollary 10. *Suppose $\mathcal{A}$ is an initialized HIOA with state models indexed by I. If $\mathcal{A}$ has a cycle with extra switches, then it has a cycle with extra switches that has fewer than $|I|^3$ mode switches.*

Theorem 11. *Suppose $\mathcal{A}$ is an initialized HIOA with state models indexed by I. For any $\tau_a > 0$, τ_a is an ADT for $\mathcal{A}$ if an only if all cycles of length at most $|I|$ are free of extra switches.*

Proof. Follows from Corollary 10 and the definition of the optimization problem $\mathsf{OPT}(\tau_a)$. ■

This theorem gives us a method for verifying ADT of initialized HIOAs by maximizing $\mathsf{OPT}(\tau_a)$ over all cycles of length at most $|I|$. In other words, for verify ADT of initialized hybrid systems it suffices to solve the optimization problem over a much smaller set of executions than we set out with at the beginning of Section 21.4. For non-initialized HIOA $\mathcal{A}$, the first part of Theorem 9 holds. That is, solving $\mathsf{OPT}(\tau_a)$ over all cycles of length at most $|I|$, if a cycle with extra switches is found, then we can conclude that τ_a is not an ADT for $\mathcal{A}$. Solving $\mathsf{OPT}(\tau_a)$ relies on formulating it as a mathematical program such that standard mathematical programming tools can be used. This is the topic of the next section.

Example: Verifying ADT. The problem of solving $\mathsf{OPT}(\tau_a)$ for $\mathtt{Aut}(G)$ reduces to checking whether G contains a cycle of length m, for any $m > 1$, with cost less than $m\tau_a$. This is the well known mean-cost cycle problem for directed graphs and can be solved in $O(|\mathcal{V}||\mathcal{E}|)$ time using Bellman-Ford algorithm or Karp's minimum mean-weight cycle algorithm [CLR90]. In particular, for LinHSwitch with $m = 3, c_1 = 2, c_2 = 4$, and $c_3 = 5$, we compute the minimum mean-cost cycle. The cost of this cycle, which is also the ADT of this automaton, is $\frac{19}{40}\log(1+h)$.

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