ECE 584: Embedded System Verification
Constructing Invariants for Hybrid Automata.

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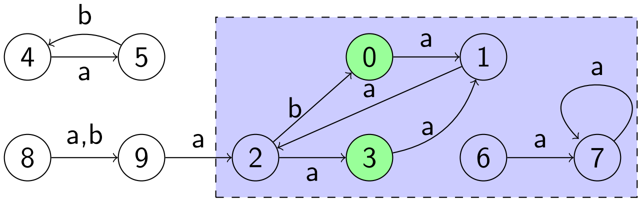
Outline

1. Invariant Synthesis
2. Hybrid Systems
3. Invariants For Hybrid Systems
4. Constraint-Based Invariant Generation

Invariants

Finite State Automaton:



Initial States: $\{0, 3\}$.
Invariant Set: Subset of states that contains

1. All initial states, and
2. All states reachable in one or more steps from initial.

Example:

$\{0, 1, 2, 3, 6, 7\}$ $\{0, 1, 2, 3, 4, 5\}$
$\{0, 1, \ldots, 9\}$ $\{0, 1, 2, 3\}$ $\leftarrow$ **Strongest Invariant**

Proving Programs Correct

Compute $\lceil \sqrt{n} \rceil$, for $n \geq 0$.

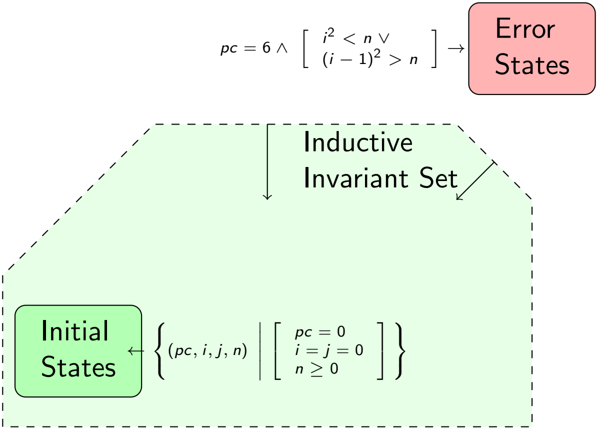
```
int computeSqrt ( int n )
    @Pre: n ≥ 0
    int i, j = (0, 0);
    while ( j ≤ n ) {
        i := i + 1;
        j := j + 2 * i − 1;
    }
    @Post: i^2 ≥ n ∧ (i − 1)^2 ≤ n
```

Infinite State Automaton

$$(pc, i, j, n) \longrightarrow (pc', i', j', n')$$

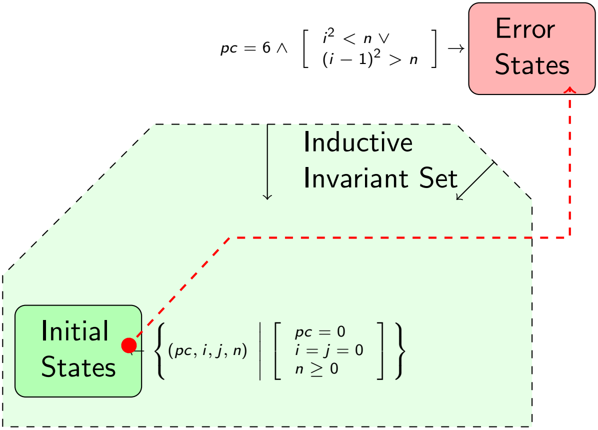
Inductive Invariant Set

Start inside invariant set $\Rightarrow$ remain in invariant set.



Inductive Invariant Set

Start inside invariant set ⇒ remain in invariant set.



Proving Programs Correct

Example: Compute $\lceil \sqrt{n} \rceil$, for $n \geq 0$.

```
int computeSqrt ( int n )
    @Pre: n ≥ 0
1:  int i, j = (0, 0);
2:  while ( j ≤ n ) {
3:      i := i + 1;
4:      j := j + 2 ∗ i − 1;
5:  }
    @Post: i^2 ≥ n ∧ (i − 1)^2 ≤ n
```

Ind. Invariant at location 2:

$$j = i^2,\ n \geq 0,\ j \leq n + 1,\ j \leq n - 2i - 1\,.$$

Q: How do we find the invariant above?

Automatic Invariant Synthesis Techniques

(Reasonably) well-understood problem for programs:

- Abstract Interpretation with Widening Framework:
  - Intervals: [Cousot+Cousot'78]
  - Polyhedral Invariants: [Cousot+Halbwachs'78]
  - Octagons [Miné'02]
  - Templates [S.+Sipma+Manna'05]
  - Symbolic Ranges [S.+Ivancic+Gupta'07]
  - Algebraic/Semi-algebraic [Carbonell+Others'05]
- Constraint-based Invariant Generation:
  - Linear Invariants: [Colón+S.+Sipma'03, S.+Sipma+Manna'04, Gulwani+Others'07, Gupta+Rybalchenko+Majumdhar'08,...]
  - Algebraic (Polynomial) Invariants: [S.+Sipma+Manna'04, Carbonell+Kapur'04, Seidl+Muller-Olm'04, Colón'04, ...]

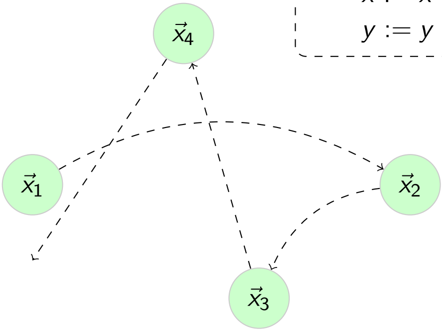
Challenge: Invariants for *Hybrid Systems*.

Dynamical Systems

Discrete dynamical systems: defined by maps.

$$\vec{x}(n+1) = F(n, \vec{x}(n))\,.$$

$$\begin{aligned}&\text{while}(x^2 + y^2 \leq 3)\\ &\qquad x := x + y^{\frac{1}{2}}\\ &\qquad y := y - x^{\frac{1}{2}}\end{aligned}$$



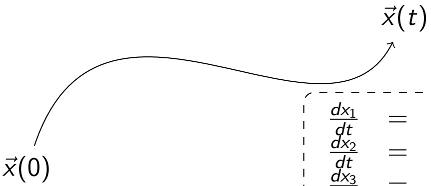
Examples:

Automata, Programs, Digital Circuits

Continuous Systems

Continuous dynamical systems: defined by flows.

$$\frac{d\vec{x}}{dt} = F(t, \vec{x})\,.$$



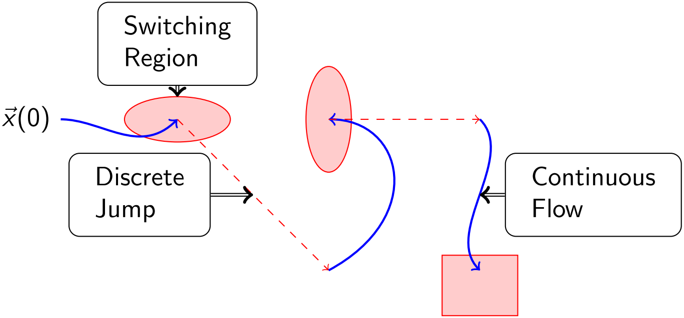
$$\begin{aligned}
\frac{dx_1}{dt} &= x_1 \sin\psi - x_2 \cos\psi \\
\frac{dx_2}{dt} &= x_1^2 - x_1 x_2 \sin(2\psi) \\
\frac{dx_3}{dt} &= cos\psi \\
\frac{d\psi}{dt} &= 0
\end{aligned}$$

Examples:

Mechanical systems, analog circuits, biological systems (cell signalling mechanisms).

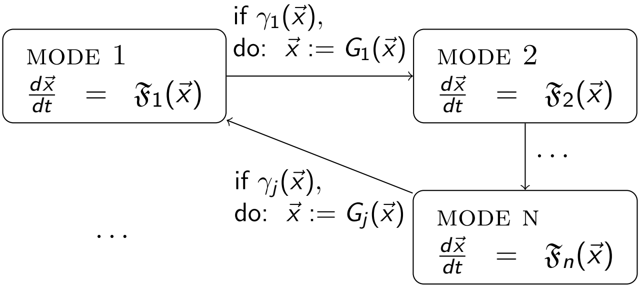
Hybrid Systems

1. Continuous-time Flows + Discrete-time Transitions.
2. *Multi-Modal*:Continuous-time dynamics depend on the state.



Hybrid Automaton

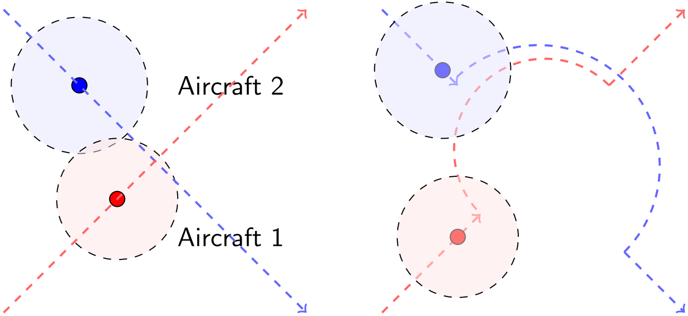
[Alur et al.'96; ...]



- Finite set of *modes*. $Q : \{q_1, \ldots, q_m\}$
- Continuous state variables. $\vec{x} : (x_1, \ldots, x_n)$.
- Dynamics for each mode.
- Discrete Transitions between modes.

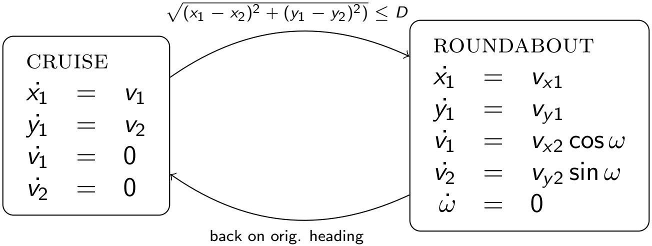
Example # 2: Conflict Resolution Maneuvers

Conflict resolution protocol. [Tomlin et al.'98]



Collision Avoidance Model

Hybrid automaton for each aircraft:



Hybrid Automata: Verification Challenges

Hybrid automata are Turing complete.

Even the simplest ones:

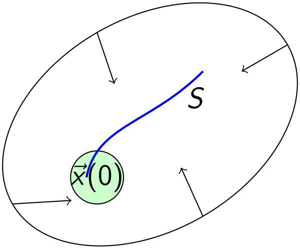
- Constant ODEs. $\frac{d\vec{x}}{dt} = \vec{c}$.
- Small number of variables ~ 3.

Verification is therefore **undecidable** even for simpler cases.

Problem: *Interesting* hybrid automata models do not fall into a known decidable class.

Invariants for Hybrid Systems.

Invariants: Differential Equations

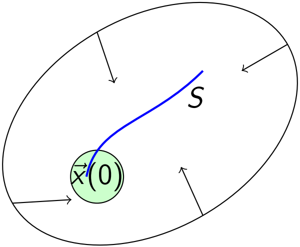


Set S is *positive invariant* for flow φ iff

$$\forall\ \vec{x}(0) \in S,\ \ \varphi(\vec{x}(0), t) \in S\,.$$

Start inside set $S \Rightarrow$ flow remains in S.

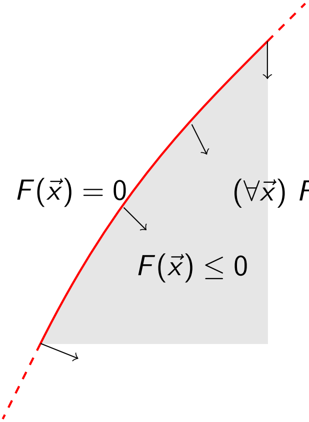
Positive Invariant Checking



Need to find *closed form* solutions.

- Not available in general for non-linear systems.
- Even for linear systems, closed form involves transcendental functions: $\sin, \cos, exp$.

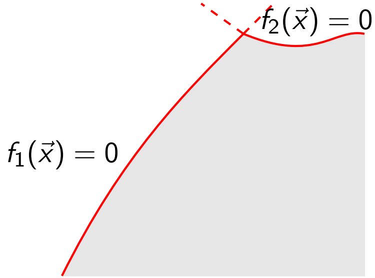
Positive Invariance Principle



$$(\forall \vec{x})\ F(\vec{x}) = 0 \ \Rightarrow\ \frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} < 0$$

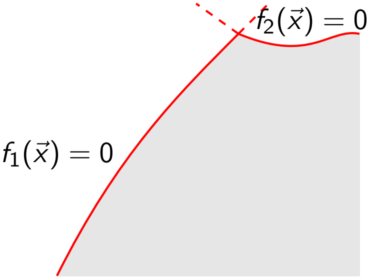
Intuition: Flow on the boundary points "inwards".
Formal Statement: Nagumo Viability Theorem/Subtangential condition.

Checking Positive Invariance



$$
\begin{array}{lll}
f_1(\vec{x}) = 0 \ \wedge \ f_2(\vec{x}) \leq 0 & \models & \frac{df_1}{dt} < 0 \\
f_1(\vec{x}) \leq 0 \ \wedge \ f_2(\vec{x}) = 0 & \models & \frac{df_2}{dt} < 0
\end{array}
$$

Checking Positive Invariance



$$f_1(\vec{x}) = 0 \;\wedge\; f_2(\vec{x}) \leq 0 \quad \models \quad \frac{df_1}{dt} < 0$$
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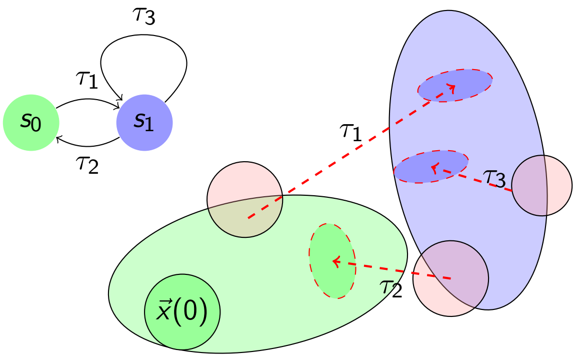
Soundness Warning: Unsound for "pathological" functions f_i.

- Restrict applications to well-behaved f_i (convex functions). [Blanchini+Miani]
- Or use a weaker version that works for corner cases. [Platzer+Clarke'07]

Invariants for Hybrid Systems

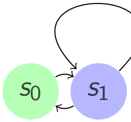


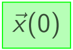
- Positive invariant for each discrete modes.
- "Preserved" by discrete transitions.
- Invariant checking reduced to checking entailments.
  Efficient decision procedures for linear systems.
  Theorem proving in the general case.

Invariant Synthesis for Hybrid Systems

Invariant Synthesis.

[Henzinger et al.'96,...]

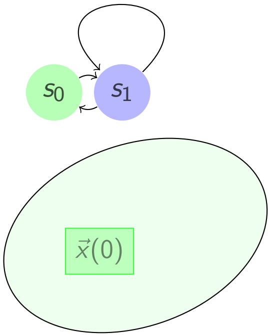




- Start with Initial States.

Invariant Synthesis.

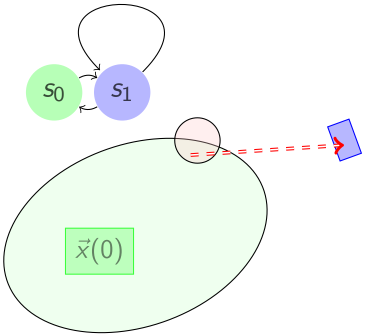
[Henzinger et al.'96,...]



- Start with Initial States.
- Positive Invariant computation for differential equations.

Invariant Synthesis.

[Henzinger et al.'96,...]



- Start with Initial States.
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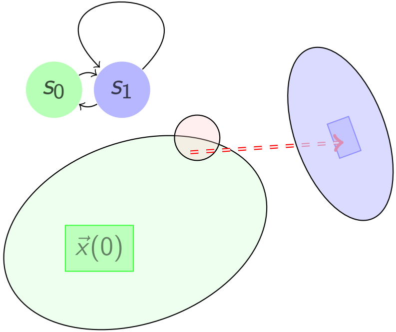

- Start with Initial States.
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- Iterate until convergence (Widening/Extrapolation)

Invariant Synthesis.

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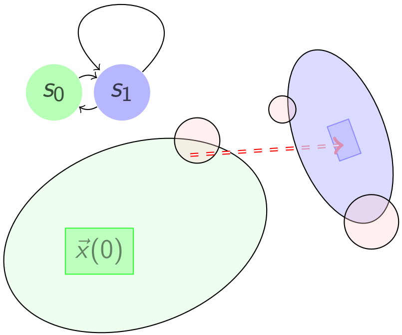

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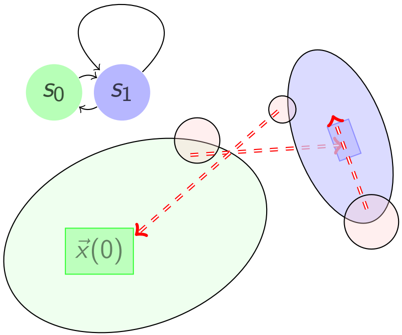

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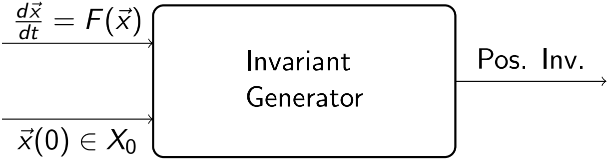
Invariants for Continuous Systems



Search for smallest set I such that:

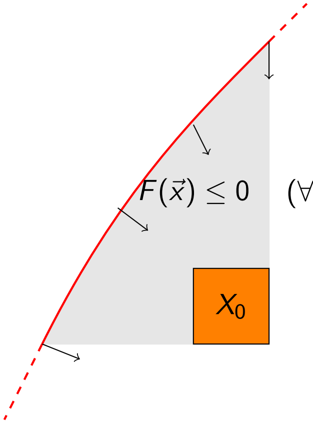
1. $X_0 \subseteq I$.
2. I is a positive invariant. For each time trajectory $\vec{x} : [0, T] \mapsto \mathcal{R}^n$, we have

$$(\vec{x}(0) \in I \;\Rightarrow\; \forall\ t \in [0, T],\ \vec{x}(t) \in I)\ .$$

Invariant Synthesis for Continuous Systems

- **Constraint-Based** techniques: [S.+others'04, Prajna+Jadbabaie'04, Gulwani+Tiwari'09, Platzer+Clarke'09,...]
  1. Fix a family of invariants by choosing an *ansatz*.
  2. Deduce constraints for invariants.
  3. Solve to obtain invariants.
- **Iterative** techniques: [S.+Others'06 - '11]
  1. Express positive invariants as fixed point of monotone operator.
  2. Use Kleene iteration to compute fixed points.
- **Flowpipe** construction techniques: [Zhao'93, Asarin+Others'00, Krogh+Others'98, Kurzhanski +Varaiya'07, Girard'05, ...]
  1. Time bounded invariants.
  2. Rigorous set-valued integration for a class of sets and ODEs.
- **Other techniques**: Optimal control theory, interval Taylor methods and many others. [Berz+Makino, Mitchell+Tomlin,...]

Constraint-Based Method



$$(\forall \vec{x})\ F(\vec{x}) = 0 \ \Rightarrow \ \frac{dF}{dt} < 0$$

Goal: Search for positive invariants F.

Constraint-Based Invariant Synthesis

1. Fix a family of candidate invariants: $F(\vec{c}, \vec{x}) \leq d$.

$$c_1 x_1 + \cdots + c_n x_n \leq d\,.$$

2. Derive constraints on $\vec{c}$ to encode positive invariance: Flow on the boundary points "inwards".

$$(\forall\ \vec{x})\ \left[\sum_i c_i x_i = d\ \Rightarrow\ \frac{d(\sum_i c_i x_i)}{dt} < 0 \right]$$

3. Solve constraints to find invariants.

Encoding Positive Invariance

Given ODE,

$$\frac{d\vec{x}}{dt} = F(\vec{x}),\ \vec{x}(0) \in X_0 \text{ and}$$

ansatz $g(\vec{c}, \vec{x}) \leq 0$, **search for** a positive invariant function

- Initial conditions

$$\forall\ \vec{x}\ (\vec{x} \in X_0)\ \Rightarrow\ g(\vec{c}, \vec{x}) \leq 0\,.$$

- Positive Invariance

$$\forall\ \vec{x}\ g(\vec{c}, \vec{x}) = 0\ \Rightarrow\ (\nabla g) \cdot F < 0\,.$$

Recall Lagrangian Relaxation for LP

- Primal: $\max c^T x$ subject to $Ax \leq b, x \geq 0$;
- Dual: $\min b^T y$ subject to $A^T y \geq c, y \geq 0$;
- **Weak Duality Theorem:** For any feasible y of the Dual and feasible x of the primal, $b^T y \geq c^T x$.
- Langrange relaxation: $\max c^T x + \lambda^T (b - Ax)$, λ's are called the Lagrange multipliers
- $\forall \hat{\lambda}, c^T x^* \leq c^T x^* + \hat{\lambda}^T (b - Ax^*) \leq c^T \bar{x} + \hat{\lambda}^T (b - A\bar{x})$
- For any fixed set of $\hat{\lambda}$ values, the optimal for the Lagrangian Relaxation will be no smaller than the optimal result to Primal.

Lagrangian Relaxation

Idea: Use Lagrange multipliers to dualize the implication:

$$(\forall\ \vec{x})\ g_1(\vec{x}) \leq 0\ \wedge\ \ldots\ \wedge\ g_m(\vec{x}) \leq 0\ \Rightarrow\ g(\vec{x}) \leq 0\,.$$

$$(\exists\ \lambda_1, \ldots, \lambda_m \geq 0)\ \lambda_1 g_1 + \cdots + \lambda_m g_m \equiv g$$

It is always sound to relax this way.

Lagrangian Relaxation

Idea: Use Lagrange multipliers to dualize the implication:

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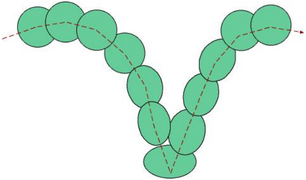
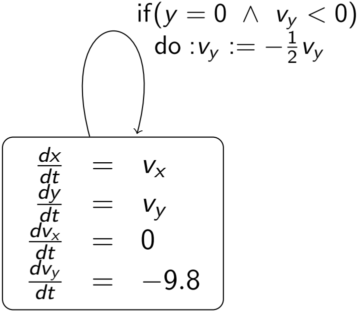
It is always sound to relax this way. *Complete* in the following cases:

1. $g_1, \ldots, g_m, g$ are affine functions (**Farkas' Lemma**).
2. $m = 1, g_1, g$ are positive-semidefinite quadratic forms (**S-Procedure**).
3. $g_1, \ldots, g_m, g$ are convex and satisfy some constraint qualifications.
4. ...

Example # 1: Bouncing Ball





Initial Conditions:

$$x(0) \in [0, 1], y(0) \in [2, 3], v_x(0) \in [1, 2], v_y(0) \in [-2, 2]$$

Ansatz

$$c_0 + c_1 x + c_2 y + c_3 v_x + c_4 v_y \leq 0$$

Encoding Initialization

Implication

$$(\forall\, x, y, v_x, v_y)\ \underbrace{\begin{bmatrix} x \in [0,1] \\ y \in [2,3], \\ v_x \in [1,2] \\ v_y \in [-2,2] \end{bmatrix}}_{\text{Initial Condition}} \Rightarrow \underbrace{c_0 + c_1 x + c_2 y + c_3 v_x + c_4 v_y \le 0}_{\text{Ansatz}}$$

Dualized Condition:

$$\left(\begin{array}{lcl} c_0 & \geq & 0\lambda_1 + 1\lambda_2 - 2\lambda_3 + 3\lambda_4 - \lambda_5 + 2\lambda_6 + 2\lambda_7 + 2\lambda_8 \\ c_1 & = & \lambda_2 - \lambda_1 \\ c_2 & = & \lambda_4 - \lambda_3 \\ c_3 & = & \lambda_6 - \lambda_5 \\ c_4 & = & \lambda_8 - \lambda_7 \\ \lambda_1, \ldots, \lambda_8 & \geq & 0 \end{array}\right)$$

Encoding Positive Invariance

Implication

$$(c_0+c_1x+c_2y+c_3v_x+c_4v_y=0) \Rightarrow \frac{d(c_0+c_1x+c_2y+c_3v_x+c_4v_y)}{dt} < 0$$

$$(c_0+c_1x+c_2y+c_3v_x+c_4v_y=0) \Rightarrow c_1v_x+c_2v_y-9.8c_4 < 0$$

Dualized Implication:

$$\begin{pmatrix} \mu c_0 & < & -9.8c_4 \\ \mu c_1 & = & 0 \\ \mu c_2 & = & 0 \\ \mu c_3 & = & c_1 \\ \mu c_4 & = & c_2 \\ \mu & \geq & 0 \end{pmatrix}$$

Combined Constraints

$$\left(\begin{array}{rcl}
c_0 & \geq & 0\lambda_1 + 1\lambda_2 - 2\lambda_3 + 3\lambda_4 - \lambda_5 + 2\lambda_6 + 2\lambda_7 + 2\lambda_8 \\
c_1 & = & \lambda_2 - \lambda_1 \\
c_2 & = & \lambda_4 - \lambda_3 \\
c_3 & = & \lambda_6 - \lambda_5 \\
c_4 & = & \lambda_8 - \lambda_7 \\
\lambda_1, \ldots, \lambda_8 & \geq & 0 \\
\mu c_0 & < & -9.8 c_4 \\
\mu c_1 & = & 0 \\
\mu c_2 & = & 0 \\
\mu c_3 & = & c_1 \\
\mu c_4 & = & c_2 \\
\mu & \geq & 0
\end{array}\right)$$

Solving Constraints

Solutions:

$$
\begin{array}{lcl}
c_0 = -2, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 1 & \rightarrow & v_y - 2 \leq 0 \\
c_0 = 1, c_1 = 0, c_2 = 0, c_3 = -1, c_4 = 0 & \rightarrow & 1 - v_x \leq 0
\end{array}
$$

Current Work

Nonlinear Systems:

1. Systems with polynomial dynamics.
2. Useful computational techniques from (semi-)algebraic geometry:

   Gröbner bases, Syzygies, Sum-of-Squared Programming
3. Ideas for linearization using F-related vector fields. [S.'11]

Relational Abstraction: Invariants that relate current state to some "future" state.

Use invariants to discretize a hybrid system. [S.+Tiwari'11]

Thank you! Questions?