# Optimal Bit Rate for State Estimation of Switched Nonlinear Systems

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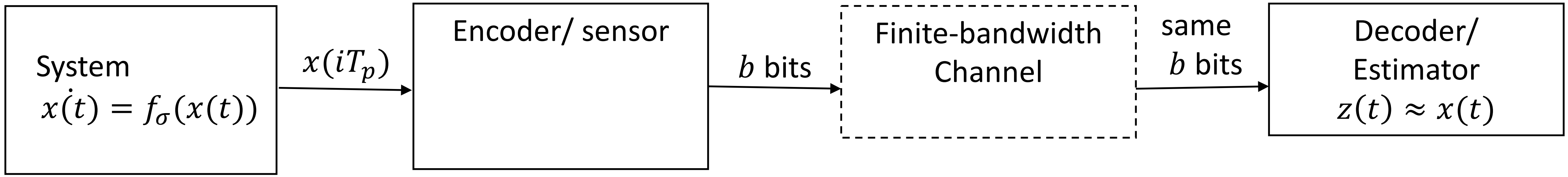
# Introduction

- More than 70 embedded computing units communicate over shared over 1 MBps CAN bus in some cars.

# Setup


System
$\dot{x}(t) = f_\sigma(x(t))$
$x(iT_p)$
Encoder/ sensor
$b$ bits
Finite-bandwidth Channel
same $b$ bits
Decoder/ Estimator
$z(t) \approx x(t)$

# Coupling between State Estimation and Model Detection

- Switching signal is not known
- State estimation and model detection problems should be tackled simultaneously

# Outline

- Introduction to state estimation
- Estimation entropy
- Upper-bound on estimation entropy
- Impossibility of estimating below entropy rates
- Estimation algorithm
  - Correctness
  - Bit Rate

# Definitions and notations

- $\dot{x} = f_\sigma(x), \sigma: [0, \infty) \to P, x \in \mathbb{R}^n, x_0 \in K$
- $|P| = N$
- $f_i$ is Lipschitz continuous with Lipschitz constant $L_i$ for each $i \in P$
  - $L = \max_{i \in P} L_i$
- $\sigma$ is unknown has a minimum dwell time $T_d$
- $\xi_\sigma(x_0, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the trajectory when the switching signal is $\sigma$ and the initial state is $x_0$

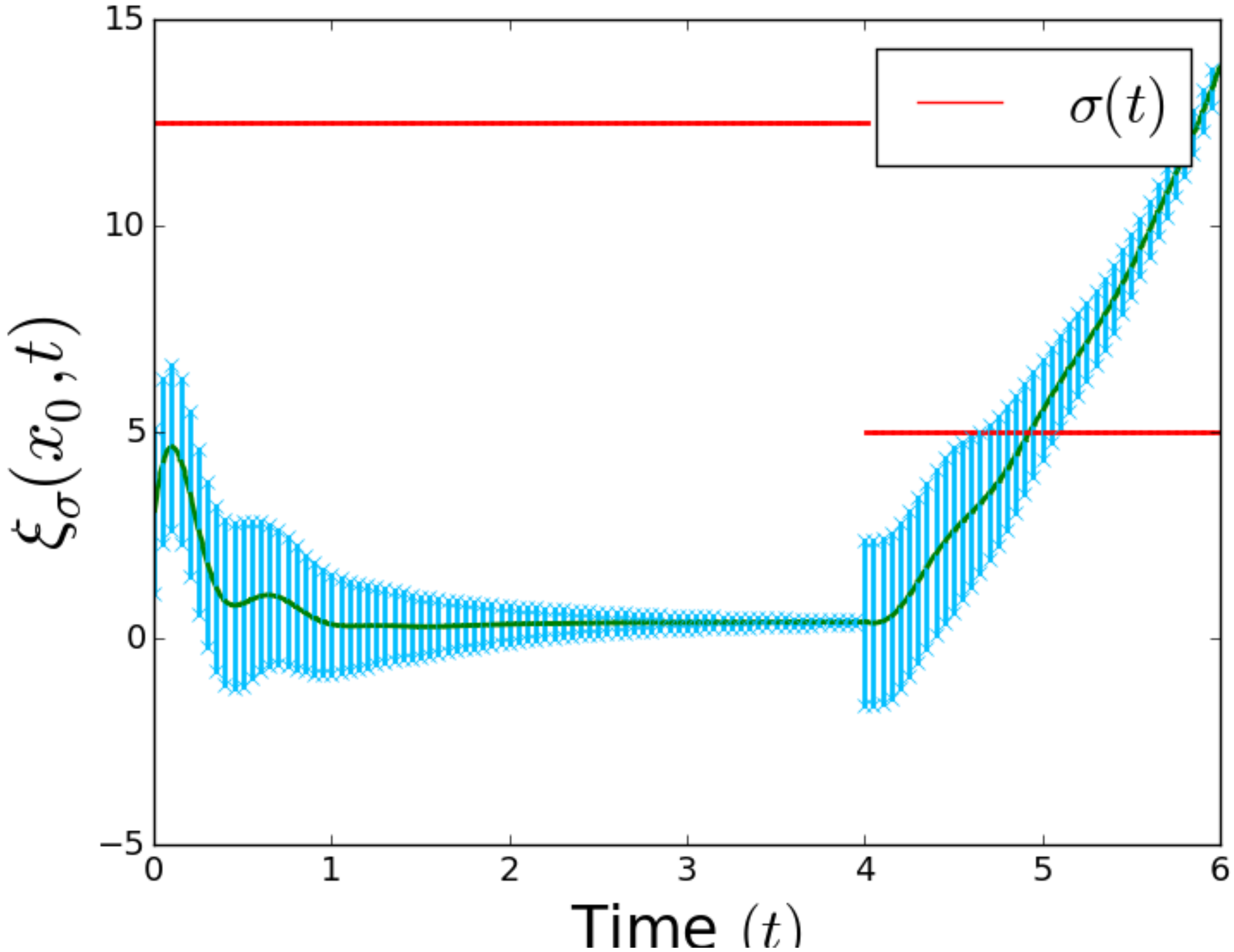
# Approximation Functions

- $z: \mathbb{R} \rightarrow \mathbb{R}^n$ is an $(\varepsilon, \alpha, \tau)$-approximating function for $\xi_\sigma(x_0, t)$ if:
  - $\|z(t) - \xi_\sigma(x_0, t)\| \leq \varepsilon$ if $t \in [s_j, s_j + \tau)$
  - $\|z(t) - \xi_\sigma(x_0, t)\| \leq \varepsilon e^{-\alpha(t-(s_j+\tau))}$ if $t \in \left[s_j + \tau, s_{j+1}\right)$

  where $s_0 = 0, s_1, \ldots$ are the switching times in $\sigma$
- Second part is similar to that of in Liberzon and Mitra HSCC'16 for nonlinear systems
  - $\|z(t) - \xi_\sigma(x_0, t)\| \leq \varepsilon e^{-\alpha t}$ if $t \geq 0$

# Approximation Functions (Continued)

Any function $z: [0, \infty) \to \mathbb{R}^n$ within
the blue intervals is $(\varepsilon, \alpha)$-approximating
function for $\xi_\sigma(x_0, t)$



# Approximation sets and Entropy

- $\hat{X} = \{\hat{x}_1, \hat{x}_2, \ldots \hat{x}_M\}$ is an $(T, \varepsilon, \alpha, \tau)$-approximating set if:

  for every $x_0 \in K$ and $\sigma \in \Sigma(T_d)$, there exists $\hat{x}_1 \in \hat{X}$ that is an $(\varepsilon, \alpha, \tau)$-approximating function for $\xi_\sigma(x_0, t)$ over $[0, T]$
- $s_{est}(T, \varepsilon, \alpha, \tau)$ is the minimum cardinality of such approximating set
- Estimation Entropy:
  - $h_{est}(\varepsilon, \alpha, \tau) = \limsup_{T \to \infty} \frac{1}{\mathrm{T}} \mathrm{s_{est}}(\mathrm{T}, \varepsilon, \alpha, \tau)$
  - [Liberzon and Mitra, HSCC'16]: $h_{est}(\alpha) = \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{\mathrm{T}} \mathrm{s_{est}}(\mathrm{T}, \varepsilon, \alpha, \mathrm{K})$

# Entropy Upper Bound

- Construct an approximating set by: designing an algorithm that constructs an approximating function for any given trajectory $\xi_\sigma(x_0, t)$
- Bound the number of functions that can be computed by the algorithm
- Substitute that number with $s_{est}$ in the definition of entropy to get:

$$h_{est}(\varepsilon, \alpha, \tau) \leq \frac{(L+\alpha)n}{\ln 2} + \frac{\log N}{T_e}$$

where $T_e$ is the largest $t$ that satisfies:

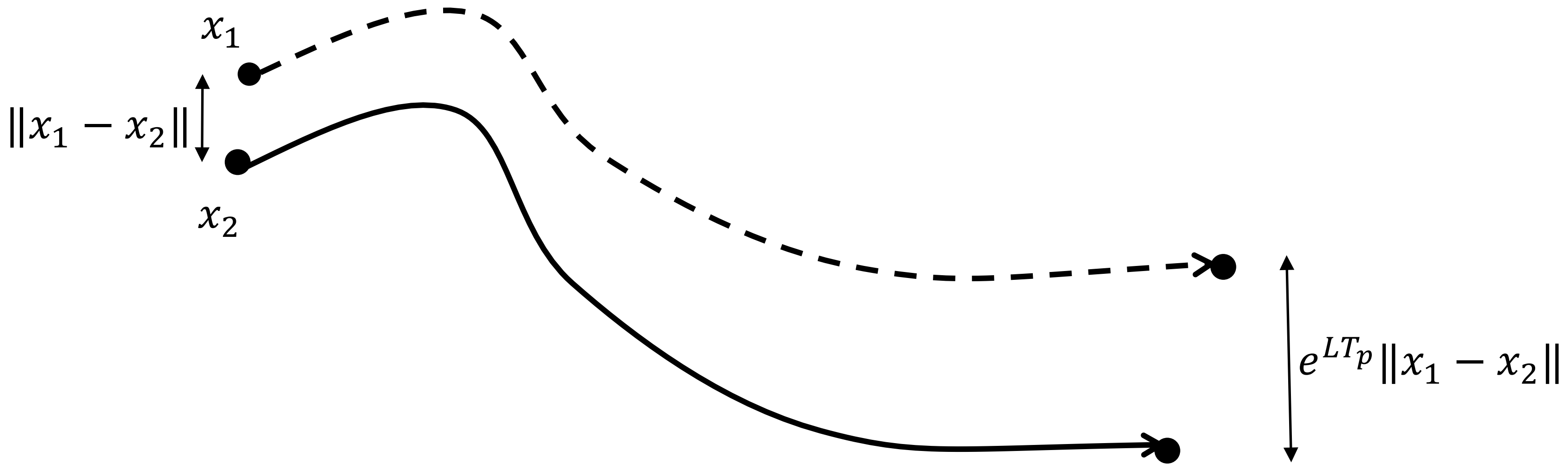
$$d(t) \leq \varepsilon(1 - e^{-\alpha(T_d - t)})$$

# Impossibility of estimating below entropy rates

- Assume $\log Q$ bits are sent each $T_p$ time units.
- Assume that the average bit rate $\left(\frac{\log Q}{T_p}\right)$ is less than $h_{est}(\varepsilon, \alpha, \tau)$, then there exists $l$ large enough s.t. $\frac{\log Q}{T_p} < \frac{1}{lT_p} \log s_{est}(lT_p, \varepsilon, \alpha, \tau)$
- Thus, $\mathrm{Q}^l < s_{est}\left(lT_p, \varepsilon, \alpha, \tau\right)$, contradiction.
  - Trajectories generated by the algorithm represents smaller approximation set.

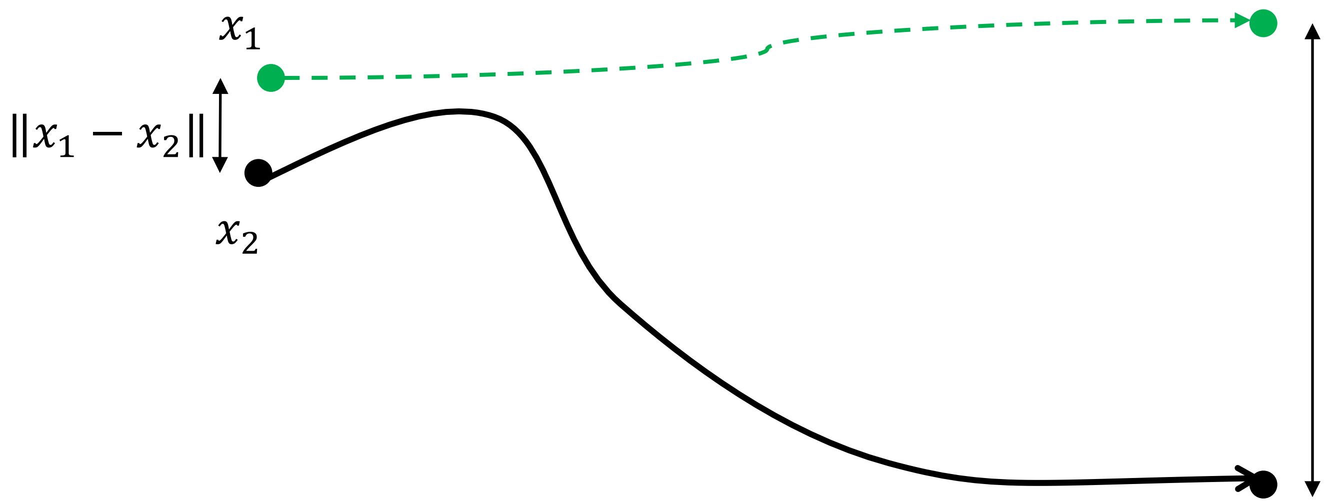
# Distance between two trajectories of the same mode after $T_p$ time units

- Same system, different initial states
- Bellman-Gronwall Inequality



# Distance between two trajectories of different modes after $T_p$ time units

- Different systems, different initial states
- Bellman-Gronwell Inequality with triangular inequality



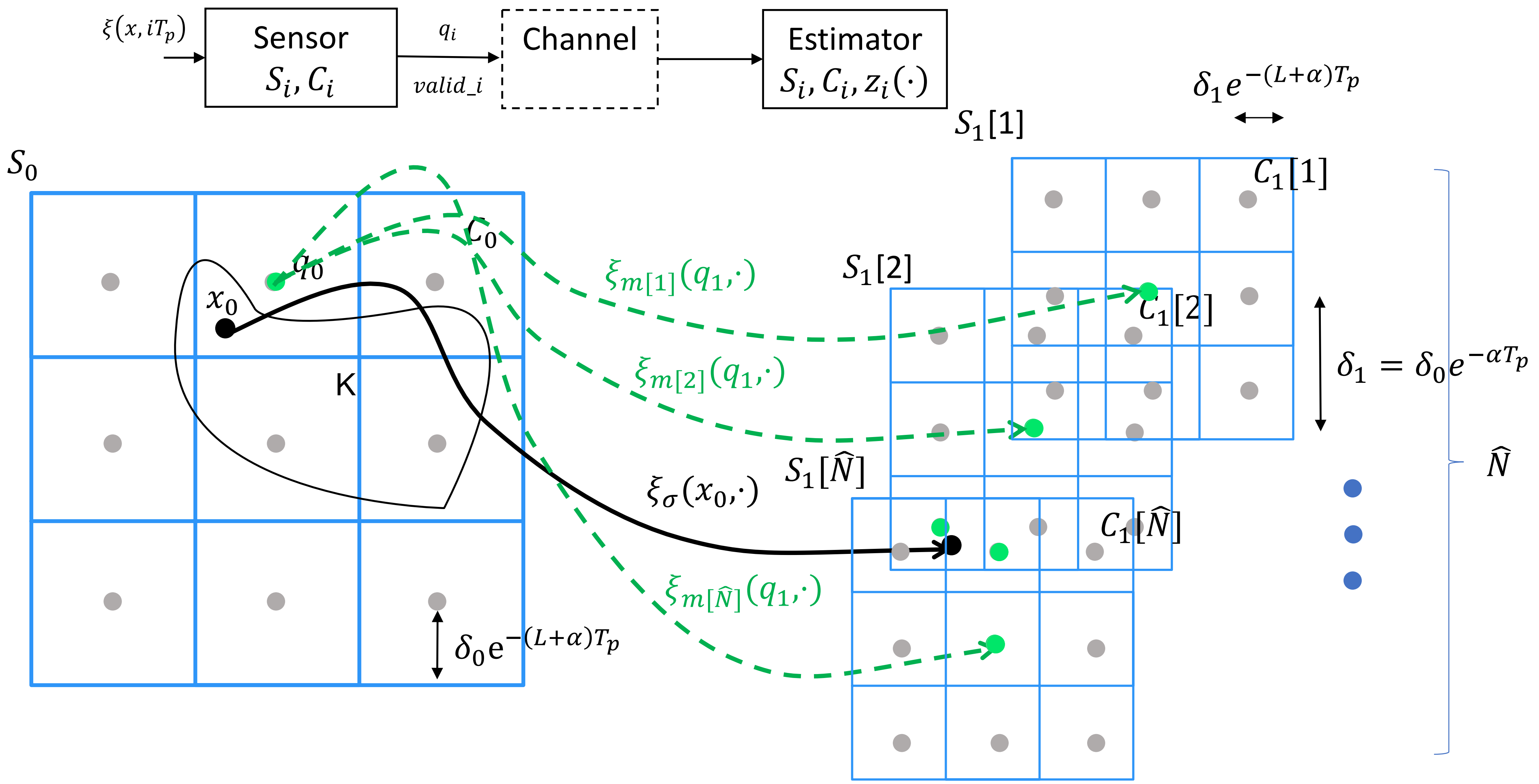

$$\leq e^{LT_p}\|x_1 - x_2\| + \int_0^{T_p} \|f_1(x_1, s) - f_2(x_2, t)\| ds$$

$$= e^{LT_p}\|x_1 - x_2\| + d(t)$$

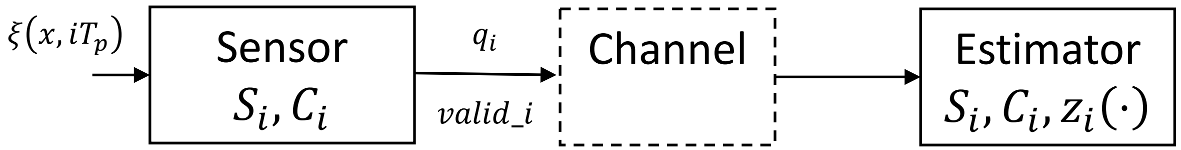
$$d(t) \coloneqq \max_{p,r \in P} \sup_{x \in Reach(K)} \int_0^t \left\| f_p\left(\xi_p(x,t)\right) - f_r\left(\xi_r(x,t)\right) \right\| dt$$

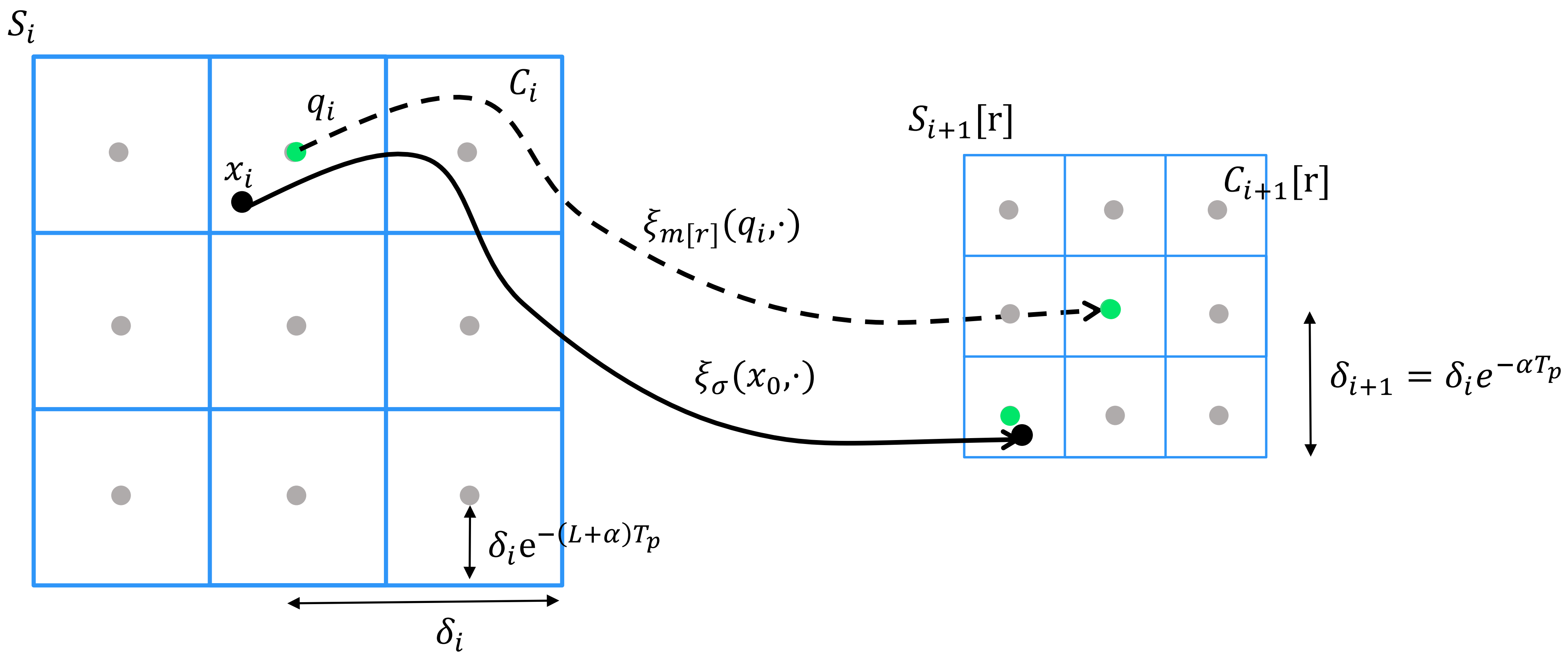
(assumed to exist for $t \leq \tau$)

# Estimation Algorithm (sensor side) (Initialization)

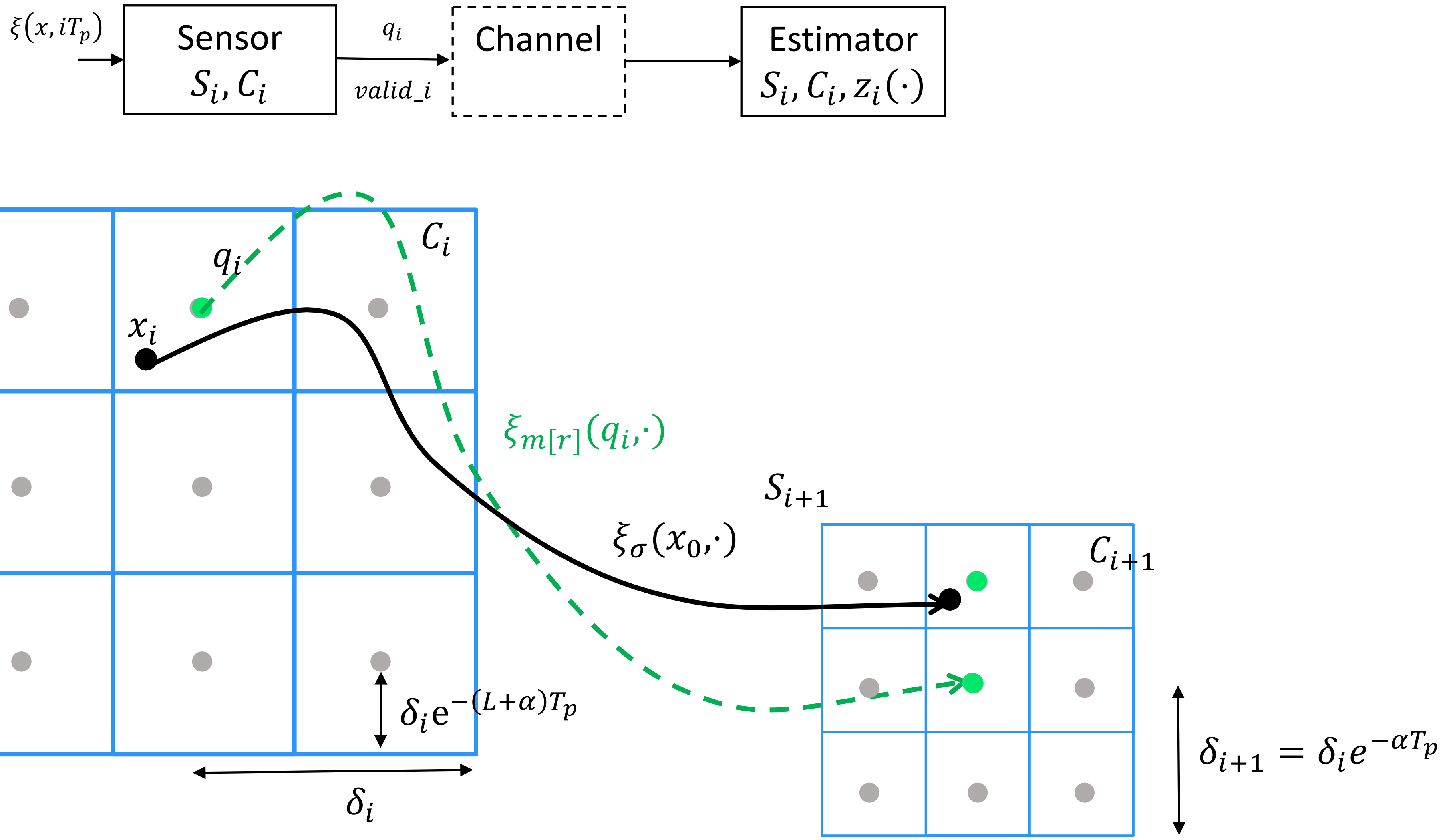


# Estimation Algorithm (correct mode ($m[r] = \sigma$), no switch)

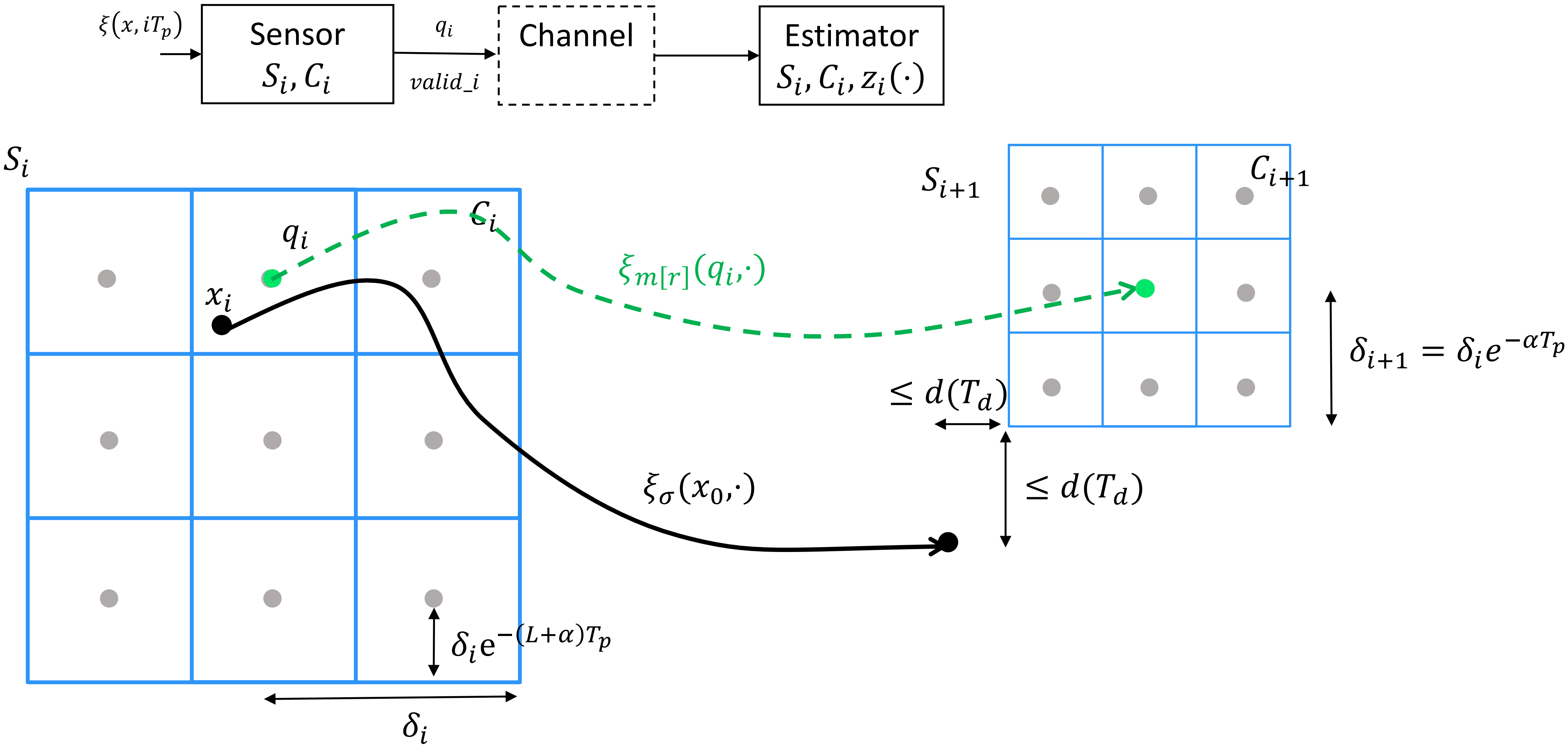


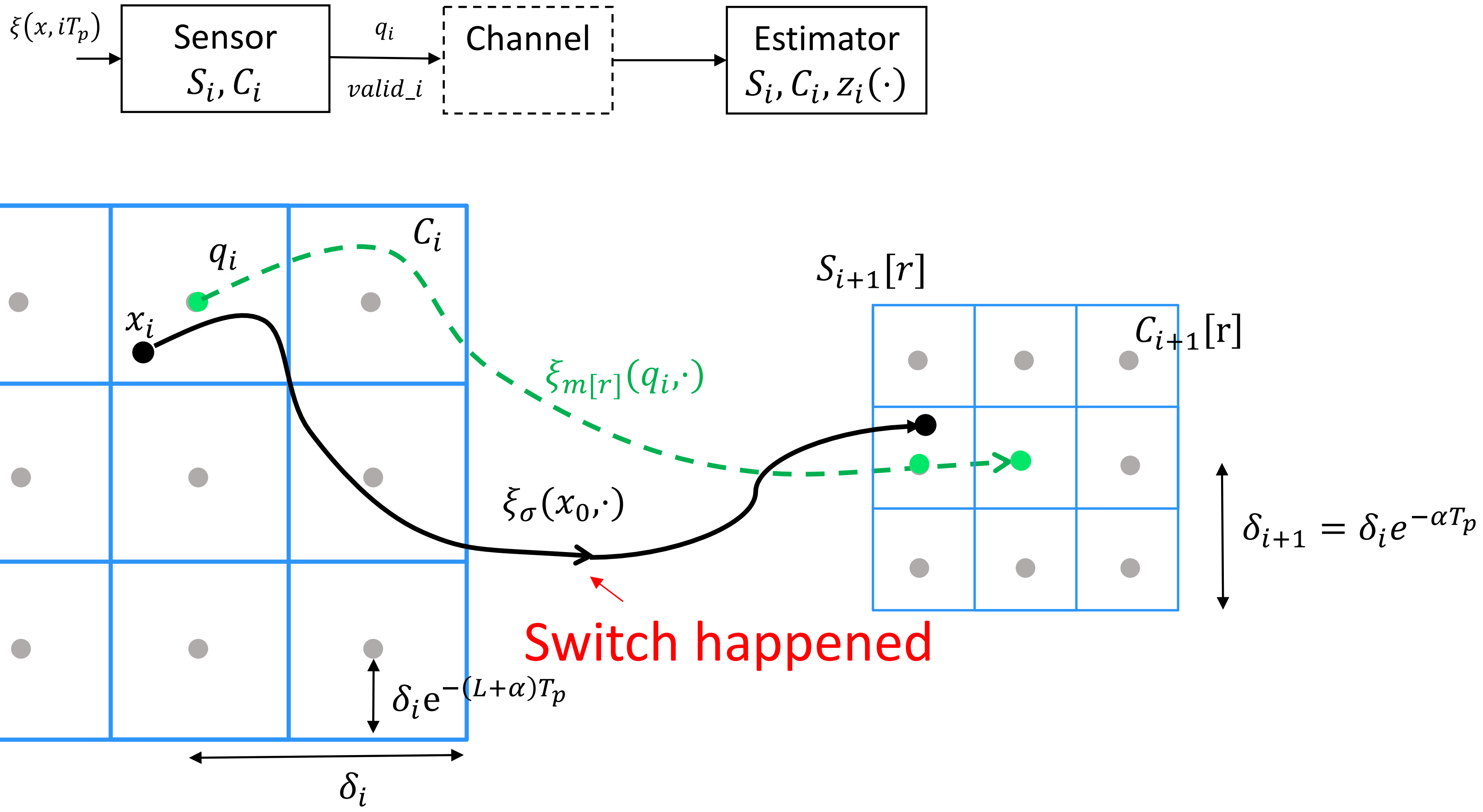
# Estimation Algorithm (wrong mode, no switch: case 1)



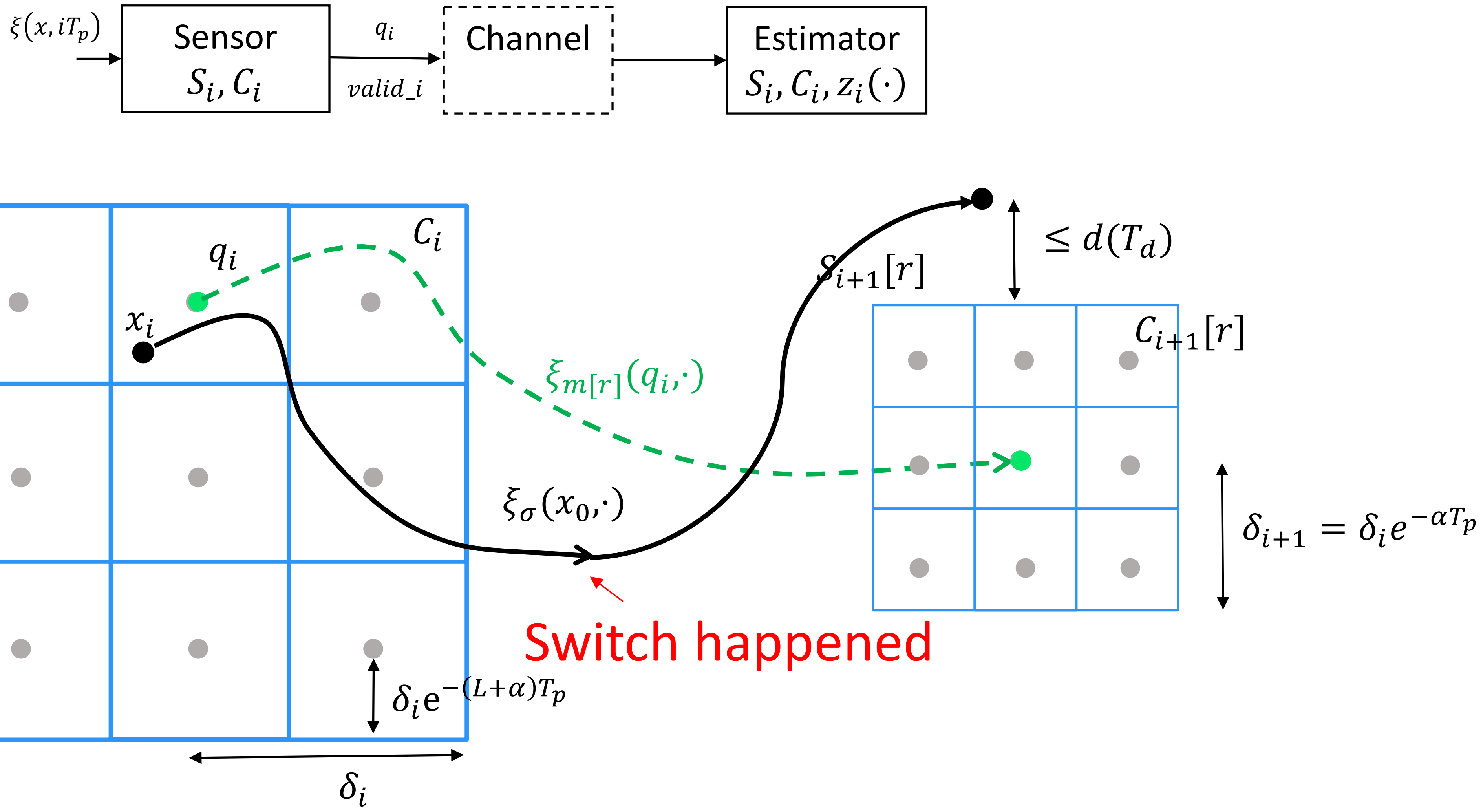
# Estimation Algorithm (wrong mode, no switch: case 2)



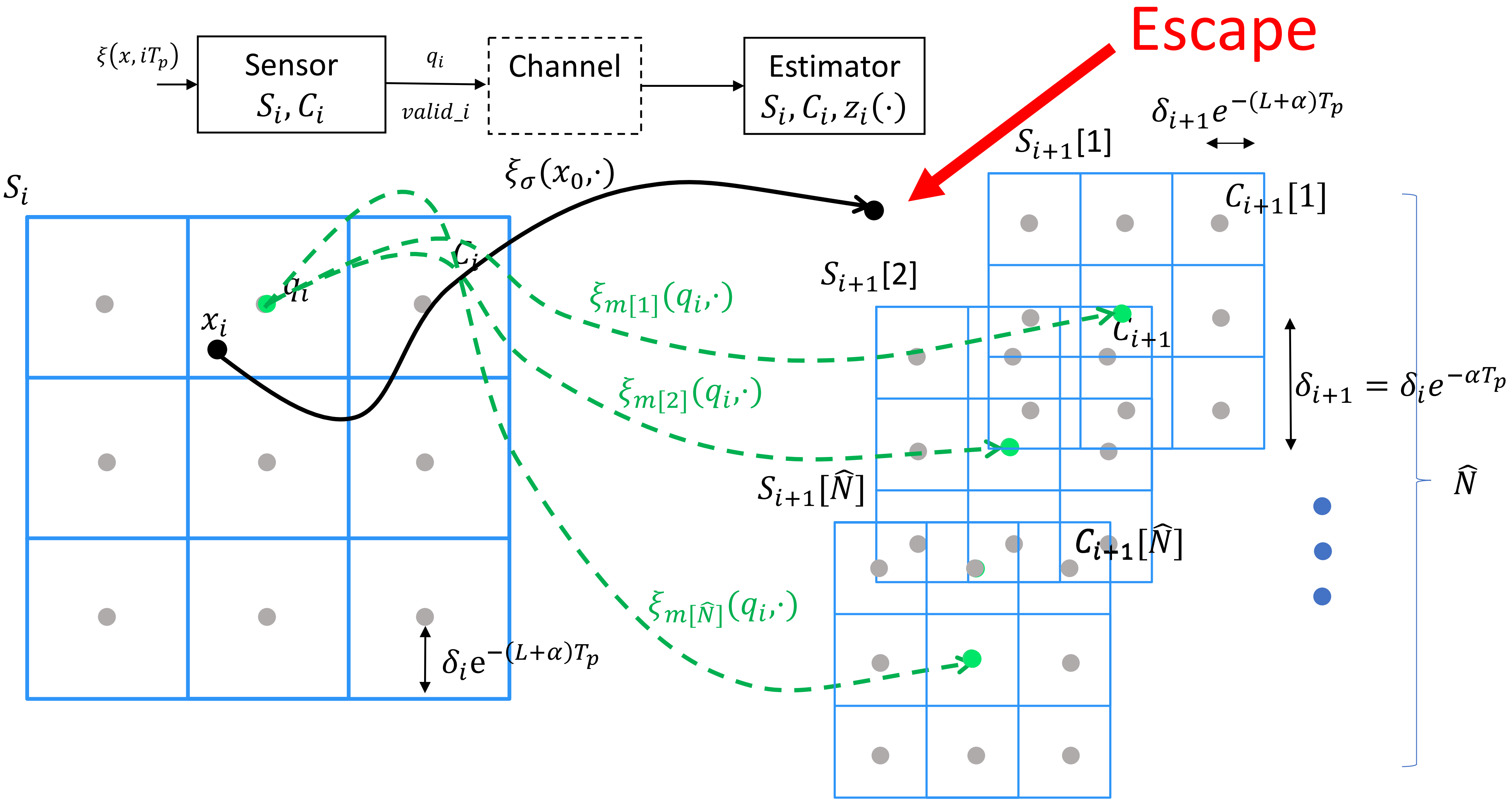
# Estimation Algorithm (switch: case 1)



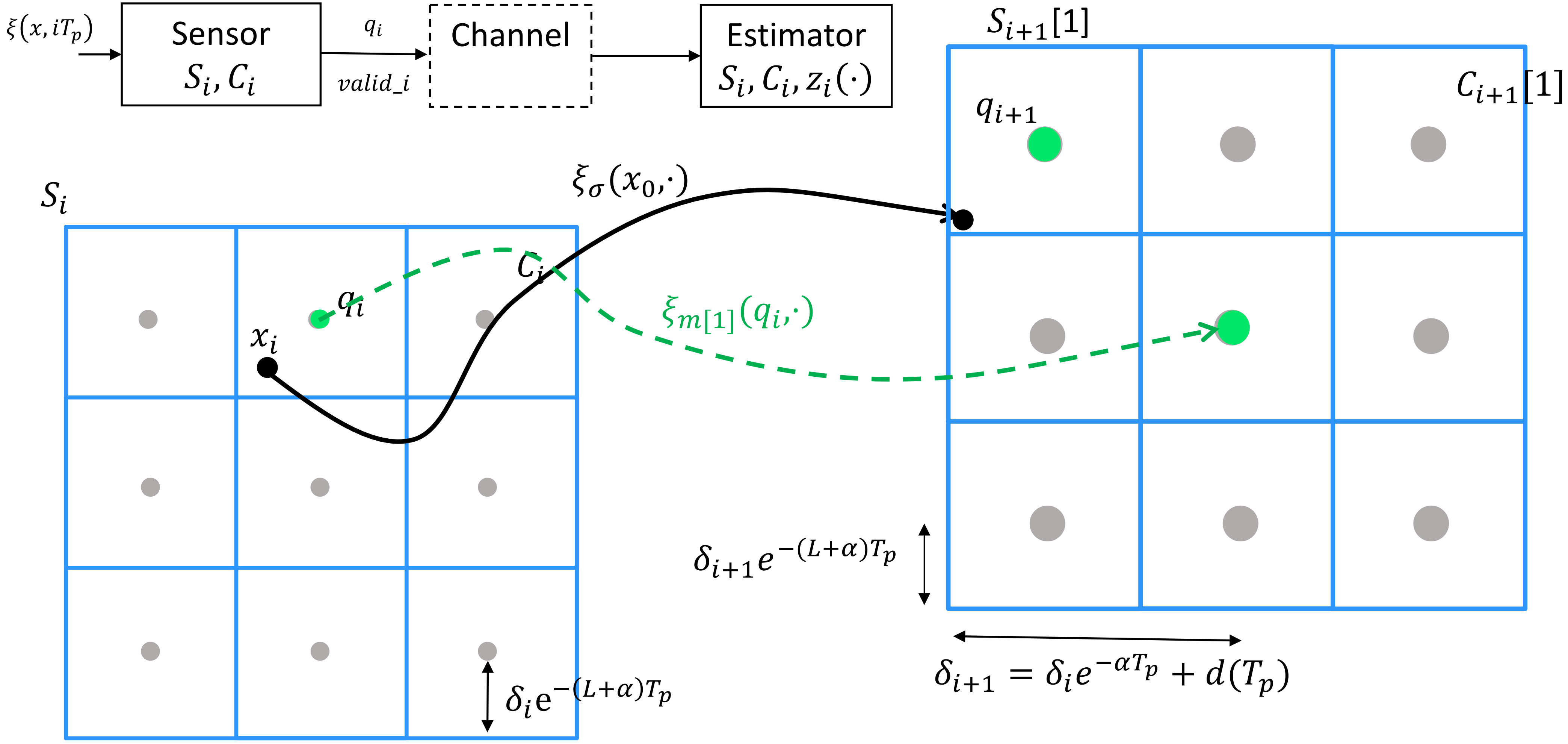
# Estimation Algorithm (switch: case 2)



# Estimation Algorithm: escapes



# Estimation Algorithm: escapes



# Estimator side Algorithm

- Same algorithm
  - It knows what modes are invalidated from the $valid$ vector
  - It knows the quantized state from the sensor

# Number of Escapes between switches

- After $\lceil \frac{N}{\widehat{N}} \rceil$ escapes all modes would have been considered
- The true mode will not be invalidated

# Exponential Separation

- We need to make sure that wrong modes will escape
- Modes p and r are $(L_s, T_s)$-exponentially separated if:

  $\exists\, \epsilon_{min} > 0$ such that for any $\varepsilon \leq \epsilon_{min}$, and for all $x_1, x_2$ with $\|x_1 - x_2\| \leq \varepsilon$,

  $$\left\|\xi_p(x_1, t) - \xi_r(x_2, t)\right\| > \varepsilon e^{L_s T_s}$$

  (definition from Liberzon and Mitra HSCC'16)

- All modes are assumed to be $\left(L, T_p\right)$-exponentially separated, unless:

  $\left\|\xi_p(x_1, t) - \xi_r(x_2, t)\right\| \leq \varepsilon e^{L T_p}$ for all $x_1$ and $x_2$ reached by the system

# Number of Iterations to get the right mode

- Bounds the number of iteration to falsify a wrong mode to:

$$i_{inv}(\delta) \coloneqq \max\{\lceil \frac{1}{\alpha T_p} \ln \frac{\delta}{\epsilon_{min}} - \frac{L}{\alpha} \rceil, 1\}$$

  when $\delta$ is the radius of $S$

- Maximum value of $\delta$ will be:

$$\delta_{max} = \max_{i \in \{1, \lceil N/\hat{N} \rceil\}} \delta_0 e^{-i\alpha T_p} + d(T_p) \frac{1 - e^{-i\alpha T_p}}{1 - e^{-\alpha T_p}}$$

  which happens when the escapes follows each other in consecutive iterations after a switch.

- Number of iterations needed to invalidate all wrong modes:

$$i_{det} \leq \left\lceil \frac{N}{\hat{N}} \right\rceil i_{inv}(\delta_{max}) + 2$$

  (coarse upper-bound)

# Estimation Theorem

- Number of iterations needed to decrease $\delta$ from $\epsilon_{min}$ to $\delta_0$ is:

$$i_{est} \coloneqq \max(\left\lceil \frac{1}{\alpha T_p} \ln \frac{\epsilon_{min}}{\delta_0} \right\rceil, 0)$$

- If minimum dwell time is greater than $(i_{det} + i_{est} + 1)T_p$, then

$$\|\xi_\sigma(x_0, t) - z(t)\| \le \begin{cases} \delta_{max} + d(T_p), \forall t \in [s_j, s_j + i_{det}T_p) \\ \delta_{max} e^{-\alpha(t - (s_j + i_{det}T_p))}, \forall t \in [s_j + i_{det}T_p, s_{j+1}) \end{cases}$$

  For any $j \in \mathbb{N}$.

- $T_p, \delta_0$ and $\widehat{N}$ can be chosen so that $z(t)$ will be an approximating function for $\xi_\sigma(x_0, t)$

# Algorithm Bit Rate

- At each iteration, it sends:
  - Bit vector $valid$ of size $\widehat{N}$
  - Quantized version of the state with respect to a grid with $\left\lceil \frac{\delta}{\delta e^{-(L+\alpha)T_p}} \right\rceil^n$ points
- So, the average bit rate is:

$$\frac{(L+\alpha)n}{\ln 2} + \frac{\widehat{N}}{T_p}$$

- Gap from the entropy upper bound:

$$\frac{\widehat{N}}{T_p} - \frac{\log N}{T_e}$$

  - Remember $T_e \leq T_p$