State Estimation of Dynamical Systems with Unknown Inputs: Entropy and Bit Rates

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ABSTRACT

Finding the minimal bit rate needed for state estimation of a dynamical system is a fundamental problem in control theory. In this paper, we present a notion of topological entropy, to lower bound the bit rate needed to estimate the state of a nonlinear dynamical system, with unknown bounded inputs, up to a constant error ε. Since the actual value of this entropy is hard to compute in general, we compute an upper bound. We show that as the bound on the input decreases, we recover a previously known bound on estimation entropy – a similar notion of entropy – for nonlinear systems without inputs [10]. For the sake of computing the bound, we present an algorithm that, given sampled and quantized measurements from a trajectory and an input signal up to a time bound $T > 0$, constructs a function that approximates the trajectory up to an ε error up to time T. We show that this algorithm can also be used for state estimation if the input signal can indeed be sensed in addition to the state. Finally, we present an improved bound on entropy for systems with linear inputs.

CCS CONCEPTS

• **Computer systems organization → Sensors and actuators**; *Robotic control*; Sensor networks;

KEYWORDS

Entropy, State Estimation, Bit Rates, Nonlinear Systems, Discrepancy Functions

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1 INTRODUCTION

State estimation is a fundamental problem for controlling and monitoring dynamical systems. In most application scenarios, the estimator has to work with plant state information sent by a sensor over a channel with finite bit rate. If a certain accuracy is required from the estimator, then a natural question is to ask: *what is the minimal bit rate of the channel for the estimator to support this accuracy requirement?* This question has been investigated for both stochastic and non-stochastic systems and channels. In the non-stochastic setting, the point of view of topological entropy has proven to be a fruitful line of investigation. In particular, it is useful for deriving the minimal necessary bit-rates for systems without inputs (see, for example [10, 11, 17]). This paper contributes in this line of investigation and proposes answers to the above question for systems with bounded, piece-wise continuous input.

Several definitions for topological entropy of control systems have been proposed and they have been related to minimal data rates necessary for controlling the system over a communication channel [9, 12, 13]. Nair et al. in [12] defined topological feedback entropy for stabilization of discrete-time systems in terms of cardinality of open covers in the state space. An alternative definition in terms of spanning open-loop control functions was proposed later by Colonius and Kawan in [2]. Equivalence between these two notions was presented in [3] and the formulation was extended to continuous-time dynamics in [2]. More in-depth discussion about the different notions of entropy was presented in [9]. Recently, Rungger and Zamani presented a notion of entropy [14] representing the needed state information by a feedback controller to keep a nondeterministic discrete time system invariant in a subset of the state space. In the context of switched systems, Yang and Liberzon [18, 19] proposed control design methodology for systems with unknown disturbance and a limited bit rate feedback channel.

From the point of view of estimation, the topological entropy of a dynamical system represents the rate at which the number of distinguishable trajectories with finite precision exponentially increase over time. In [10, 11], Liberzon and Mitra introduced a notion of estimation entropy for continuous-time nonlinear systems, defined in terms of the number of system trajectories that approximate all other trajectories up to an exponentially decaying error. They also considered a second notion of estimation entropy which uses approximating functions that are not necessarily trajectories of the system, and they established the equivalence of these two notions. They then showed that it is impossible to perform state estimation with bit-rates lower than entropy. While computing entropy exactly for a given system is generally hard, [10] showed that we can upper-bound it by $\frac{n(L_x+\alpha)}{\ln 2}$. Here n is the dimension of the system and L_x is the Lipschitz constant of the vector field. In [16], Schmidt presented bounds on the topological entropy of switched linear systems with Lie structures. In our previous paper [17], we extended the notion of estimation entropy to switched nonlinear systems

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while providing upper bounds and a state estimation algorithm. None of these works consider models with inputs.

In this paper, we study the problem of estimating the state of nonlinear dynamical systems with unknown, possibly discontinuous, inputs. This is a more challenging problem because even if the uncertainty about the state can be made to decrease over time using sensor measurements, the uncertainty about the input may not decrease. The input can change arbitrarily with few constraints and the continuous effect of the uncertain input prevents the uncertainty about the state from going to zero. We contend this using a weaker notion of estimation, akin to that in [15], that only requires the error to be bounded by a constant $\varepsilon > 0$, instead of exponentially decaying down to zero.

We show that there is no state estimation algorithm with a bit rate smaller than the entropy. For the purpose of computing an upper bound, we use a corrected version of a previous result in [8] to upper bound the sensitivity of a trajectory of a nonlinear system to changes in the initial state and in the input signal. Then, we present a procedure that, given sampled states of a trajectory and corresponding sampled values of an input signal, constructs a function that estimates the trajectory. This procedure is of independent interest, as it can also be used as an estimation algorithm provided the unknown input signal can be sampled. We count the number of trajectories that can be constructed by this procedure for different initial states and input signals, up to a time bound T. The rate of exponential growth of this number as T increases gives an upper bound on entropy.

The upper bound is presented in terms of the state and the input dimensions n and m, global bounds on the norm of the Jacobian matrices of the vector field with respect to the state and the input, M_x and M_u, the upper bound on the norm of the input u_{max}, and two constants μ and η that represent how much the input signal is allowed to vary over time. Roughly, η upper bounds the size of the jumps in the input signal and μ constrains the number of large jumps in a short amount of time. We show that if the upper bound on the input norm goes to zero, the upper bound on estimation entropy would be $\frac{n(nL_x+1/2)}{\ln 2}$, only a $O(n)$ factor larger than the one computed in [10] for α equal to zero. The entropy upper bound increases logarithmically with u_{max} and quadratically with η when ε is small. It also increases as $O(\varepsilon^{-2})$ as the allowed error ε decreases.

Finally, we compute an upper bound on entropy of systems with linear inputs. We present a more precise way to compute the sensitivity of the system with respect to changes in the initial state and in the input signal. We show how our results can be used to get sufficient bit rates for state estimation for two examples.

The paper is organized as follows: we start with a few preliminaries in Section 2. We define the system along with its entropy in Section 3. Then, we compute the upper bound on entropy for general nonlinear systems in Section 4. After that, we compute a new upper bound on entropy for systems where the input affects the dynamics linearly in Section 5. Finally, we discuss the results and suggest future directions in Section 6.

2 PRELIMINARIES

Vector norms and covers. For a real vector $v \in \mathbb{R}^n$, we denote by $\|v\|$ the infinity norm of the vector and by v^T the transpose of v. $B(v, \delta)$ is a δ-ball—closed hypercube of radius δ—centered at v. For a hyperrectangle $S \subseteq \mathbb{R}^n$ and $\delta > 0$, $grid(S, \delta)$, is a collection of 2δ-separated points along axis parallel planes such that the δ-balls around these points cover S. In that case, we say that the grid is of size δ. For a compact set $S \subset \mathbb{R}^n$, $diam(S) = \max_{x_1, x_2 \in S} \|x_1 - x_2\|$ denotes its diameter. We denote by $[a; b]$, the set of integers in $\mathbb{Z}$ that belong to the interval $[a, b]$. For a matrix A, $\lambda_{max}(A)$ denotes the largest eigenvalue of A. Note that for any positive definite matrix A, $\lambda_{max}(A) \leq |||A|||$, where $|||\cdot|||$ is any matrix norm. For a finite set S, we denote by $|S|$ the cardinality of S. We say that a continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$.

Definition 1. *Let U be a compact set in $\mathbb{R}^m$ and $u_{max} > 0$ be the maximum infinity norm of a vector in U. Fix μ and $\eta \geq 0$ and let $\mathcal{U}(\mu, \eta, u_{max})$ be the set of all piecewise-right-continuous functions u that map $\mathbb{R}_{\geq 0}$ to U and satisfy:*

$$\|u(t + \tau) - u(t)\| \leq \mu\tau + \eta, \quad (1)$$

for all t and $\tau \geq 0$.

Geometrically, Definition 1 means that for any $u \in \mathcal{U}(\mu, \eta, u_{max})$, and any $t \in \mathbb{R}_{\geq 0}$, and any $\tau \geq 0$, $u(t + \tau)$ should belong to the truncated m-dimensional cone with radius η and rate of divergence μ as shown in Figure 1.

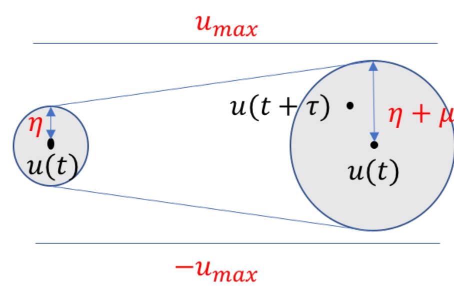

Figure 1: Constraints on the variation of u

This constraint is similar to Assumption 1 in [6] which was made on the variation of the system matrix of a time-varying linear dynamical system to relate its stability conditions to those of a switched linear dynamical system with slow switching. Also, it is similar to the slow switching assumption made by Hesphana and Morse in [7] to prove the stability of switched systems with stable subsystems.

We will fix u_{max}, μ and η throughout the paper. Let u be a function in $\mathcal{U}(\mu, \eta, u_{max})$. Then, for all $t \in \mathbb{R}_{\geq 0}$, define the right and left hand limits of u at t as follows:

$$u(t^+) = \lim_{s \to t^+} u(s) \text{ and } u(t^-) = \lim_{s \to t^-} u(s).$$

If t is a point of discontinuity, we define $u(t) = u(t^+)$.

Note that for any $u_{max} > 0$, and for any set of piecewise continuous functions that map time to the corresponding U, there always exists η and μ that satisfy inequality (1) for all functions u in the set. For example, with $\eta = 2u_{max}$, the bound is satisfied trivially for any $\mu \geq 0$. In other words, for $\eta = 2u_{max}$, u can have many points of discontinuity (jumps), in a short interval, each with a difference between the before and after values being as large as having a norm of $2u_{max}$. On the other hand, having $\eta = 0$, means that all the functions in $\mathcal{U}(\mu, \eta, u_{max})$ are Lipschitz continuous with constant μ. In general, knowing that u cannot vary much, i.e. having few points of discontinuity or small gradient, can be expressed by setting μ and η to smaller values. η restricts the maximum norm of a jump and μ restricts the number of large jumps in a short interval.

3 NON-AUTONOMOUS DYNAMICAL SYSTEMS AND ENTROPY

We consider a dynamical system of the form:

$$\dot{x}(t) = f(x(t), u(t)), \tag{2}$$

where $t \in \mathbb{R}_{\geq 0}$, the initial state x_0 is in a compact set $K \subset \mathbb{R}^n$, $u \in \mathcal{U}(\mu, \eta, u_{max})$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The function f is globally Lipschitz with constants L_x and L_u, and piecewise-continuous Jacobian matrices $J_x = \frac{\partial f(x,u)}{\partial x}$ and $J_u = \frac{\partial f(x,u)}{\partial u}$, with respect to the first and second arguments, respectively.

Once an initial state x_0 and an input function $u \in \mathcal{U}(\mu, \eta, u_{max})$ are fixed, the solution exists and is unique. We denote it by $\xi_{x_0,u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$.

Given a time bound $T > 0$, initial state $x_0 \in K$ and an input signal $u \in \mathcal{U}(\mu, \eta, u_{max})$, we say that a function $z : [0, T] \to \mathbb{R}^n$ is *ε-approximating* for the trajectory $\xi_{x_0,u}$ over the interval $[0, T]$, if

$$\|z(t) - \xi_{x_0,u}(t)\| \leq \varepsilon, \tag{3}$$

for all $t \in [0, T]$. We say that a set of functions $\mathcal{Z} := \{z \mid z : [0, T] \to \mathbb{R}^n\}$ is (T, ε, K)-approximating for system (2), if for every $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$, there exists an ε-approximating function $z \in \mathcal{Z}$ for the trajectory $\xi_{x_0,u}$ over $[0, T]$. The minimal cardinality of such a set is denoted by $s_{\text{est}}(T, \varepsilon, K)$.

The entropy of system (2) is defined as follows:

$$h_{\text{est}}(\varepsilon, K) = \limsup_{T \to \infty} \frac{1}{T} \log s_{\text{est}}(T, \varepsilon, K). \tag{4}$$

It represents the exponential growth rate in the number of distinguishable trajectories of the system. Hence, it also represents the bit rate need to be sent by the sensor so that the estimator can construct a "good" estimate of the state.

Note that the upper bounds on entropy we derive in the following sections approach infinity as ε approaches zero. This suggests that the entropy may not stay finite as ε approaches zero. Hence, we do not take the limit as ε goes to zero in the definition of entropy as is common in the existing literature [10].

3.1 Relation between entropy and the bit rate of estimation algorithms

In this section, we show that there is no state estimation algorithm for system (2) that requires a fixed bit rate smaller than its entropy. First, let us define state estimation algorithms which guarantee that the estimation error is bounded $\varepsilon > 0$:

Definition 2. *A state estimation algorithm for system (2) with a fixed bit rate is a pair of functions $(\mathcal{S}, \mathcal{E})$, where $\mathcal{S} : \mathbb{R}^n \times Q_s \to \Gamma \times Q_s$, $\mathcal{E} : \Gamma \times Q_e \to ([0, T_p] \to \mathbb{R}^n) \times Q_e$, T_p is the sampling time, Γ is an alphabet with N symbols, for some $N \in \mathbb{N}$, and Q_s and Q_e are the sets of internal states of the sensor $\mathcal{S}$ and estimator $\mathcal{E}$, respectively.*

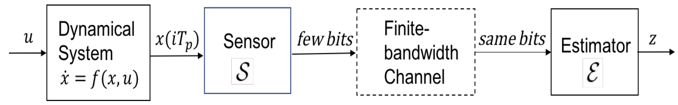

Figure 2: Block diagram showing the flow of information from the system to the sensor to the estimator.

$\mathcal{S}$ runs at the sensor side and $\mathcal{E}$ on the estimator one. $\mathcal{S}$ samples the state of the system each T_p time units and sends to $\mathcal{E}$ a symbol from Γ representing an estimate of the state at the corresponding sampling time. Finally, $\mathcal{E}$ maps the received symbol to an ε-approximating function of the trajectory for the next T_p time units (see Figure 2).

Now, let us define the bit rate of the algorithm:

$$b_r(\varepsilon, K) := \limsup_{T \to \infty} \frac{1}{T} \sum_{i=0}^{\lfloor T/T_p \rfloor} \log N = \frac{\log N}{T_p}.$$

Proposition 1. *There is no state estimation algorithm for system (2) with a fixed bit rate smaller than its entropy.*

Proof. The proof is similar to the proof of Proposition 2 in [17]. For the sake of contradiction, assume that there exists such an algorithm with a bit rate smaller than $h_{\text{est}}(\varepsilon, K)$. Recall that $h_{\text{est}}(\varepsilon, K) = \limsup_{T\to\infty} 1/T \log s_{\text{est}}(T, \varepsilon, K)$. Then, for a sufficiently large T', we should have $\frac{(l+1)\log N}{T'} < \frac{1}{T'} \log s_{\text{est}}(T', \varepsilon, K)$, where $l = \lfloor T'/T_p \rfloor$. Hence, we get the inequality $N^{l+1} < s_{\text{est}}(T', \varepsilon, K)$. However, N^{l+1} is the number of possible sequences of symbols of length $l + 1$ that can be sent by the sensor over $l + 1$ iterations. There are $l + 1$ instead of l iterations over the interval $[0, T']$ since the sensor starts sending the codewords at $t = 0$ s. Hence, the number of functions that can be constructed by the estimator is upper bounded by N^{l+1}. Moreover, for any given trajectory of the system, the output of the estimator is a corresponding ε-approximating function over the interval $[0, T']$. This is true since the estimator should be able to construct an ε-approximating function for the corresponding trajectory of the system over the interval $[0, (l + 1)T_p)$ given the codewords sent by the sensor in the first $l + 1$ iterations. Hence, the set of functions that can be constructed by the estimator defines a (K, ε, T')-approximating set. But, $s_{\text{est}}(T', \varepsilon, K)$ is the minimal cardinality of such a set. Therefore, the set of functions that can be constructed by the algorithm is a (T', ε, K)-approximating set with a cardinality smaller than s_{est}, the supposed minimal one. □

4 ENTROPY UPPER BOUND AND ALGORITHM

In this section, we derive an upper bound on the entropy of system (2) in terms of its parameters and the required bound on the estimation error, ε. We will need to first upper bound the distance between any two trajectories of the system in terms of the distance between the initial states and that between the input signals (Section 4.1). Then, in Section 4.2, we will describe a procedure that, given $\varepsilon > 0$, a time bound $T > 0$, an initial state $x_0 \in K$ and an input $u \in \mathcal{U}(\mu, \eta, u_{max})$, constructs an ε-approximating function for the trajectory $\xi_{x_0,u}$ over the interval $[0, T]$. We will count the number of functions that can be produced by this procedure for any fixed ε and T (and varying $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$) to upper bound the cardinality of the minimal approximating set. This will be used to derive the upper bound on entropy in Section 4.3.

4.1 Local input-to-state discrepancy function construction

We use a modified version of the definition of local input-to-state discrepancy as introduced in [8] in order to upper bound the distance between any two trajectories. We relax the original definition to include piece-wise continuous input signals and piece-wise continuous Jacobian matrices.

Definition 3. *(Local IS Discrepancy). For system (2), a function $V : \mathcal{X}^2 \to \mathbb{R}_{\geq 0}$ is a* local input-to-state discrepancy function *over a set $\mathcal{X} \subset \mathbb{R}^n$ and a time interval $[t_0, t_1] \subset \mathbb{R}_{\geq 0}$ if:*

(i) there exist class-$\mathcal{K}$ functions $\bar{\alpha}, \underline{\alpha}$ such that for any $x, x' \in K$, $\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \bar{\alpha}(\|x - x'\|)$, and

(ii) there exist a class-$\mathcal{K}$ function in the first argument $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class-$\mathcal{K}$ function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any $x_0, x_0' \in \mathcal{X}$ and $u, u' \in \mathcal{U}(\mu, \eta, u_{max})$, if $\xi_{x_0, u}(t), \xi_{x_0', u'}(t) \in \mathcal{X}$ for all $t \in [t_0, t_1]$, then for any such t,

$$V(\xi_{x_0', u'}(t), \xi_{x_0, u}(t)) \leq \beta(\|x_0 - x_0'\|, t - t_0) + \int_{t_0}^{t} \gamma(\|u(s) - u'(s)\|) ds. \tag{5}$$

The local discrepancy function V together with β and γ give the sensitivity of the trajectories of the system to changes in the initial state and the input. The functions $\bar{\alpha}, \underline{\alpha}, \beta, \gamma$ are sometimes called witnesses of the local IS discrepancy V. Techniques for computing local discrepancy functions have been presented [4, 5, 8]. Here we correct and use a result [8] (with proof in the Appendix) for the sake of completeness.

The following is a correction and straight-forward generalization of Lemma 15 of [8] to handle systems with piece-wise continuous inputs (instead of continuous ones).

Lemma 1. *The function $V(x, x') := \|x - x'\|^2$ is a local IS discrepancy for system (2) over any compact set $\mathcal{X} \subset \mathbb{R}^n$ and interval $[t_0, t_1] \subseteq \mathbb{R}_{\geq 0}$, with*

$$\beta(y, t - t_0) := e^{2a(t - t_0)} y^2 \text{ and } \gamma(y) := b^2 e^{2a(t_1 - t_0)} y^2,$$

where $t \in [t_0, t_1]$,

$$a := \sup_{\substack{t \in [t_0, t_1] \\ u \in \mathcal{U}(\mu, \eta, u_{max}), x \in \mathcal{X}}} \lambda_{max}\Big(\frac{J_x + J_x^T}{2}\Big) + \frac{1}{2} \text{ and}$$
$$b := \sup_{\substack{t \in [t_0, t_1] \\ u \in \mathcal{U}(\mu, \eta, u_{max}), x \in \mathcal{X}}} |||J_u|||. \tag{6}$$

Since f is globally Lipschitz continuous in both arguments, one can infer that a and b are finite over all the input and state spaces. We will denote a global upper bound on a by M_x and on b by M_u. An example of such bounds is presented in the following proposition, with the proof being in the Appendix.

Proposition 2. *For any time interval $[t_0, t_1] \subset \mathbb{R}_{\geq 0}$ and compact set $\mathcal{X} \subset \mathbb{R}^n$, $a \leq nL_x' + \frac{1}{2}$ and $b \leq m\sqrt{m}L_u'$, where L_x' and L_u' are the Lipschitz constants of f with respect to each coordinate of the state and the input respectively.*

Therefore, for any $\tau > 0$, $t \in [0, \tau]$, $x_0, x_0' \in \mathbb{R}^n$, and $u, u' \in \mathcal{U}(\mu, \eta, u_{max})$, the squared distance between the trajectories of $\xi_{x_0, u}$ and $\xi_{x_0', u'}$, $\|\xi_{x_0, u}(t) - \xi_{x_0', u'}(t)\|^2$, is upper bounded by:

$$e^{2M_x t}\|x_0 - x_0'\|^2 + M_u^2 e^{2M_x \tau} \int_0^t \|u(s) - u'(s)\|^2 ds. \tag{7}$$

Further if f has a continuous Jacobian, one can get tighter local bounds on a and b that depend on the set of input functions $\mathcal{U}(\mu, \eta, u_{max})$, the compact set $\mathcal{X}$, and the interval $[t_0, t_1]$ [4].

4.2 Approximating set construction

Let us fix $\varepsilon > 0$ throughout this section. We will describe a procedure (Algorithm 1) that, given a time bound $T > 0$, an initial state $x_0 \in K$ and an input signal $u \in \mathcal{U}(\mu, \eta, u_{max})$, constructs an ε-approximating function for the trajectory $\xi_{x,u}$ over the time interval $[0, T]$. It follows that the set of functions that can possibly be constructed by that procedure for different $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$ is a (T, ε, K)-approximating set for system (2). An upper bound on its cardinality will give an upper bound on entropy in the next section.

Algorithm 1 Construction of ε-approximating function.

input: T,T_p, δ_x, δ_u
$S_0 \leftarrow K$;
$C_{x,0} \leftarrow$ $grid(S_0, \delta_x)$;
$C_u \leftarrow$ $grid(U, \delta_u)$;
$i \leftarrow 0$;
while $i \leq \lfloor \frac{T}{T_p} \rfloor$ **do**
    $x_i \leftarrow \xi_{x_0, u}(iT_p)$;
    $q_{x,i} \leftarrow quantize(x_i, C_{x,i})$;
    $q_{u,i} \leftarrow quantize(u(iT_p), C_u)$;
    $z_i \leftarrow \xi_{q_{x,i}, q_{u,i}}$;
    i++; {parameters for next iteration}
    $S_i \leftarrow B(z_{i-1}(T_p{}^-), \varepsilon)$;
    $C_{x,i} \leftarrow grid(S_i, \delta_x)$;
    wait(T_p);
end while
output: $\{z_i : 0 \leq i \leq \lfloor \frac{T}{T_p} \rfloor\}$

The procedure (Algorithm 1) is parameterized by a time horizon $T > 0$, a sampling period $T_p > 0$, two quantization constants δ_x and $\delta_u > 0$. The procedure also uses the initial set K, the input set U, and a particular initial state $x_0 \in K$ and an input $u \in \mathcal{U}(\mu, \eta, u_{max})$ for system (2). The output is a piece-wise continuous function $z : [0, T] \to \mathbb{R}^n$ that is constructed over each $[iT_p, (i+1)T_p)$ interval for $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$. Later we will infer several constraints on the parameters such that the output z is indeed an ε-approximating function for the given trajectory $\xi_{x_0, u}$.

Initially, S_0 is set to be the initial set K. $C_{x,0}$ is a grid of size δ_x over K and C_u is a grid of size δ_u over U. At the i^{th} iteration, $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$, x_i stores the value $\xi_{x_0, u}(iT_p)$. Then, $q_{x,i}$ is set to be the quantization of x_i with respect to $C_{x,i}$. Similarly, $q_{u,i}$ is set to be the quantization of $u(iT_p)$ with respect to C_u. With slight abuse of notation, we will also denote the function of time that maps the interval $[0, T_p)$ to $q_{u,i}$ by $q_{u,i}$, as in line 10, for example. The variable z_i stores the trajectory that results from running system (2) starting from initial state $q_{x,i}$, with input signal $q_{u,i}$, and running for T_p time units. After that, i is incremented by 1 and the next iteration variables S_i and $C_{x,i}$ are initialized. Finally, the procedure outputs the concatenation of the z_i's, for all $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$ that is denoted later by the function $z : [0, T] \to \mathbb{R}^n$.

In the following lemma, we show that if the parameters of the procedure T_p, δ_x and δ_u, are small enough, then the output is an ε-approximating function for $\xi_{x_0,u}$.

Lemma 2. *Fix a constant $k \in (0, 1)$ and the parameters T_p, δ_x, and δ_u, such that:*

(1) $\varepsilon\sqrt{k} \geq \delta_x e^{M_x T_p}$, and

(2) $\varepsilon\sqrt{1-k} \geq M_u e^{M_x T_p} \sqrt{\frac{1}{3}\mu^2 {T_p}^3 + (\delta_u + \eta)\mu {T_p}^2 + (\delta_u + \eta)^2 T_p}$.

Then, for any $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$, for all $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$, and for all $t \in [iT_p, (i+1)T_p)$,

(i) $x_i \in S_i$,

(ii) $\|z_i(t - iT_p) - \xi_{x_i,u_i}(t - iT_p)\| \leq \varepsilon$,

where $u_i(t) := u(iT_p + t)$, the i^{th} piece of the input signal of size T_p.

Proof. First, fix $t \in [0, T]$ and let $i = \lfloor \frac{t}{T_p} \rfloor$. Then,

$$\begin{aligned}
&\|\xi_{x_i,u_i}(t - iT_p) - \xi_{q_{x,i},q_{u,i}}(t - iT_p)\|^2 \\
&\leq \|x_i - q_{x,i}\|^2 e^{2M_x(t-iT_p)} + M_u^2 e^{2M_x T_p} \int_{iT_p}^{t} \|u(s) - q_{u,i}\|^2 ds \\
&\qquad \text{[by (7)]} \\
&\leq \|x_i - q_{x,i}\|^2 e^{2M_x(t-iT_p)} \\
&\quad + M_u^2 e^{2M_x T_p} \int_{iT_p}^{t} \Big(\|u_i(0) - q_{u,i}\| + \|u(s) - u_i(0)\|\Big)^2 ds \\
&\qquad \text{[by triangular inequality]} \\
&\leq \delta_x^2 e^{2M_x(t-iT_p)} \\
&\quad + M_u^2 e^{2M_x T_p} \int_{iT_p}^{t} \Big(\delta_u^2 + 2\delta_u \|u(s) - u_i(0)\| + \|u(s) - u_i(0)\|^2\Big) ds,
\end{aligned} \tag{8}$$

where the last inequality follows from the fact that $\|u(iT_p) - q_{u,i}\| \leq \delta_u$, $\|x_i - q_{x,i}\| \leq \delta_x$. But, we know from (1) that there exist μ and η such that for all $u \in \mathcal{U}(\mu, \eta, u_{max})$, $\|u(s) - u(iT_p)\| \leq \mu(s - iT_p) + \eta$. Hence, $\int_{iT_p}^{t} \|u(s) - u_i(0)\| ds \leq \int_{iT_p}^{t} (\mu(s - iT_p) + \eta) ds = \frac{\mu}{2}(t - iT_p)^2 + \eta(t - iT_p) \leq \frac{\mu}{2}{T_p}^2 + \eta T_p$, since $t - iT_p \leq T_p$. Similarly, $\int_{iT_p}^{t} \|u(s) - u_i(0)\|^2 ds \leq \int_{iT_p}^{t} (\mu^2(s - iT_p)^2 + 2\mu\eta(s - iT_p) + \eta^2) ds \leq \frac{1}{3}\mu^2 {T_p}^3 + \mu\eta {T_p}^2 + \eta^2 T_p$. Substituting this in (8) leads to:

$$\begin{aligned}
&\|\xi_{x_i,u_i}(t - iT_p) - \xi_{q_{x,i},q_{u,i}}(t - iT_p)\|^2 \\
&\leq \delta_x^2 e^{2M_x T_p} + M_u^2 e^{2M_x T_p} \frac{1}{3}\mu^2 {T_p}^3 + (\delta_u + \eta)\mu {T_p}^2 + (\delta_u + \eta)^2 T_p \\
&\leq k\varepsilon^2 + (1-k)\varepsilon^2 = \varepsilon^2,
\end{aligned}$$

where the last inequality follows by substituting δ_x, δ_u and T_p by their upper bounds stated in the statement of the lemma. Hence, for any $t \in [0, T]$, for $i = \lfloor \frac{t}{T_p} \rfloor$, $\|z_i(t - iT_p) - \xi_{x_i,u_i}(t)\| \leq \varepsilon$. Therefore, for all $i \in [1; \lfloor \frac{T}{T_p} \rfloor]$ and $t \in [0, T]$, $x_i \in B(z_{i-1}(T_p), \varepsilon) = S_i$. □

Corollary 1. *Under the same conditions of Lemma 2, for all $t \in [0, T]$,*

$$\|z(t) - \xi_{x_0,u}(t)\| \leq \varepsilon. \tag{9}$$

Now that we proved that, for a given trajectory $\xi_{x_0,u}$, the output of Algorithm 1 is an ε-approximating function, one can conclude that the set of all functions that can be constructed by Algorithm 1 for any input trajectory $\xi_{x_0,u}$, where $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$, is a (T, ε, K)-approximating set. Therefore, in the following lemma, we will compute an upper bound on the number of these functions to obtain upper bound on $s_{\text{est}}(T, \varepsilon, K)$.

Before stating the lemma, note that whenever we choose k, we fix $\delta_x = \varepsilon\sqrt{k}e^{-M_x T_p}$ from now on, to simplify the presentation.

Lemma 3. *For fixed $T \geq 0$, $k \in (0, 1)$, and δ_u and T_p that satisfy the conditions of Lemma 2, the number of functions that can be constructed by Algorithm 1 for all possible $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$, is upper bounded by:*

$$\begin{aligned}
&|C_{x,0}|(|C_{x,1}||C_u|)^{\lfloor \frac{T}{T_p} \rfloor + 1} \\
&\leq \lceil \frac{diam(K)}{2\varepsilon\sqrt{k}e^{-M_x T_p}} \rceil^n \left(\lceil \frac{1}{\sqrt{k}} e^{M_x T_p} \rceil^n \lceil \frac{u_{max}}{\delta_u} \rceil^m \right)^{\left(\lfloor \frac{T}{T_p} \rfloor + 1\right)}.
\end{aligned}$$

Proof. To construct an ε-approximating function for a given trajectory $\xi_{x,u}$, at an iteration $i \in [0; \lfloor \frac{t}{T_p} \rfloor]$, Algorithm 1 picks one point in $C_{x,i}$ and picks one point in C_u for each of the $\lfloor T/T_p \rfloor + 1$ iterations. Hence, the number of different outputs that it can produce is upper bounded by:

$$|C_u|^{\lfloor T/T_p \rfloor + 1} \prod_{i=0}^{\lfloor T/T_p \rfloor} |C_{x,i}|. \tag{10}$$

Now, note that $K \subseteq B(v_c, diam(K))$, for some $v_c \in \mathbb{R}^n$. Hence, in each of the n dimensions in the state space, we should partition a segment of length $diam(K)$ to smaller segments of size $2\delta_x = 2\sqrt{k}\varepsilon e^{-M_x T_p}$ to construct the grid $C_{x,0}$. Then, $|C_{x,0}| \leq \lceil \frac{diam(K)}{2\sqrt{k}\varepsilon e^{-M_x T_p}} \rceil^n$. Similarly, for all $i > 0$, $S_i = B(z_{i-1}({T_p}^-), \varepsilon)$. Hence, $|C_{x,i}| \leq \lceil \frac{2\varepsilon}{2\sqrt{k}\varepsilon e^{-M_x T_p}} \rceil^n = \lceil \frac{1}{\sqrt{k}} e^{M_x T_p} \rceil^n$, since $diam(S_i) = 2\varepsilon$. In each of the m dimensions, $u(t)$ is bounded between $-u_{max}$ and u_{max}. Hence, $diam(U) = 2u_{max}$ and $|C_u| \leq \lceil \frac{u_{max}}{\delta_u} \rceil^m$. Substituting these values in (10) leads to the upper bound in the lemma. □

4.3 Entropy upper bound

The following proposition gives an upper bound on the entropy of system (2) in terms of k, T_p and δ_u. This form provides an an intermediate level bound where the parameters of Algorithm 1 directly appear in its expression, before providing the more complex upper bound that depends directly on the system parameters. It shows the effect of our choice of the parameters of Algorithm 1. It will also help us recover the bound on estimation entropy of systems with no inputs in [10] in Corollary 2. Moreover, it provides insights about the choices of the parameters that simplify the bound.

Proposition 3. *For a fixed $k \in (0,1)$, T_p and δ_u that satisfy the conditions in Lemma 2, the entropy $h_{est}(\varepsilon, K)$ of system (2) is upper bounded by:*

$$\frac{nM_x}{\sqrt{k}\ln 2} + \frac{n}{T_p}\log(1+\sqrt{k}e^{-M_xT_p}) + \frac{m}{T_p}\log\lceil\frac{u_{max}}{\delta_u}\rceil.$$

Proof. We substitute the upper bound on the cardinality of the minimal approximating set obtained in the previous section in definition (4) to get:

$$\begin{aligned} h_{\text{est}}(\varepsilon, K) &= \limsup_{T\to\infty}\frac{1}{T}\log s_{\text{est}}(T,\varepsilon) \\ &\leq \limsup_{T\to\infty}\frac{1}{T}\log|C_{x,0}|(|C_{x,1}||C_u|)^{\lfloor\frac{T}{T_p}\rfloor+1} \\ &\qquad\qquad [\text{by Lemma 3}] \\ &\leq \limsup_{T\to\infty}\frac{1}{T}\log(\lceil\frac{1}{\sqrt{k}}e^{M_xT_p}\rceil^n\lceil\frac{u_{max}}{\delta_u}\rceil^m)^{(\lfloor\frac{T}{T_p}\rfloor+1)} \\ &\qquad\qquad [|C_{x,0}| \text{ is constant}] \\ &= \limsup_{T\to\infty}\frac{1+T_p/T}{T_p}n\log\lceil\frac{1}{\sqrt{k}}e^{M_xT_p}\rceil \\ &\qquad + \limsup_{T\to\infty}\frac{1+T_p/T}{T_p}m\log\lceil\frac{u_{max}}{\delta_u}\rceil \\ &\leq \frac{n}{T_p}\log\lceil\frac{1}{\sqrt{k}}e^{M_xT_p}\rceil + \frac{m}{T_p}\log\lceil\frac{u_{max}}{\delta_u}\rceil \\ &\leq \frac{nM_x}{\sqrt{k}\ln 2} + \frac{n}{T_p}\log(1+\sqrt{k}e^{-M_xT_p}) + \frac{m}{T_p}\log\lceil\frac{u_{max}}{\delta_u}\rceil. \end{aligned}$$

□

We show that if the bound on the input norm is negligible, we recover the upper bound on estimation entropy of $\frac{nL_x}{\ln 2}$ derived in [10] with the only difference being the replacement of L_x by M_x (which is upper bounded by $nL_x + 1/2$).

Corollary 2. *Given any $\varepsilon > 0$, $\lim_{u_{max}\to 0} h(\varepsilon, K) \leq \frac{nM_x}{\ln 2}$.*

Proof. (Sketch) First, we will fix η to $2u_{max}$ and μ to zero. Recall that setting them to these values satisfies inequality (1). Let k be approximately equal to 1. Moreover, fix δ_u to be equal to u_{max}. Doing this will set the last term ($\log\lceil\frac{u_{max}}{\delta_u}\rceil$) in the bound in Proposition 3 to zero. Recall that we also fixed δ_x to be equal to $\varepsilon\sqrt{k}e^{-M_xT_p}$. Now, observe that there exists a $T_p > 0$ that would satisfy the conditions in Lemma 2. Hence, by Proposition 3, $h_{\text{est}}(\varepsilon, K) \leq \frac{nM_x}{\sqrt{k}\ln 2} + \frac{n}{T_p}\log(1+\sqrt{k}e^{-M_xT_p})$. Moreover, as u_{max} decreases to zero, η and δ_u go to zero. Hence, the conditions of Lemma 2 become satisfied with larger values of T_p. This would result in a negligible second term in the bound which in turn results in an upper bound of $\frac{nM_x}{\sqrt{k}\ln 2}$ which is almost $\frac{nM_x}{\ln 2}$. □

The following proposition presents an upper bound on the entropy of system (2). We assume, without loss of generality, that μ and $\eta > 0$. That is not a restrictive choice since, for a given μ and η that satisfy (1), any larger values would still satisfy it.

Proposition 4. *Fix $\varepsilon > 0$, $k \in (0,1)$ and $\delta_u \in (0, u_{max}]$ and let*

$$\rho(k,\delta_u) = \Big(\frac{\delta_u+\eta}{\mu}\Big)\Bigg(-1+\sqrt[3]{1+\Big(\frac{\varepsilon}{M_ue}\Big)^2\frac{3\mu(1-k)}{(\delta_u+\eta)^3}}\Bigg). \quad (11)$$

Then, the entropy of system (2) is upper bounded by:

$$\frac{nM_x}{\sqrt{k}\ln 2} + \frac{1}{\min\{\rho(k,\delta_u), 1/M_x\}}\Big(n\log(1+\sqrt{k}) + m\log\lceil\frac{u_{max}}{\delta_u}\rceil\Big).$$

For example, k and δ_u can be $1/2$ and η, respectively.

Proof. To prove this result, it is sufficient to show that assigning

$$T_p = \begin{cases}\rho(k,\delta_u), & \text{if } \rho(k,\delta_u) \leq 1/M_x \text{ and} \\ 1/M_x, & \text{otherwise,}\end{cases}$$

satisfies condition (2) in Lemma 2. Then, the result will follow from plugging in this value in Proposition 3. First, assume that T_p is less than or equal to $1/M_x$. Then, $e^{M_xT_p}$ in condition (2) in Lemma 2 can be upper bounded by e. Thus, condition (2) is satisfied if the value of T_p satisfies the following 3rd order polynomial inequality:

$$T_p{}^3 + 3\Big(\frac{\delta_u+\eta}{\mu}\Big)T_p{}^2 + 3\Big(\frac{\delta_u+\eta}{\mu}\Big)^2T_p - 3(1-k)\Big(\frac{\varepsilon}{\mu M_ue}\Big)^2 \leq 0. \quad (12)$$

The only real root of the polynomial on the LHS is:

$$\Big(\frac{\delta_u+\eta}{\mu}\Big)\Bigg(-1+\sqrt[3]{1+\Big(\frac{\varepsilon}{M_ue}\Big)^2\frac{3\mu(1-k)}{(\delta_u+\eta)^3}}\Bigg) = \rho(k,\delta_u). \quad (13)$$

Thus, we set T_p to $\rho(k,\delta_u)$, as it is the largest value that satisfies the needed condition. If $\rho(k,\delta_u) > \frac{1}{M_x}$, assigning T_p to $\frac{1}{M_x}$ would still satisfy the conditions of Lemma 2, hence the bound. □

The following corollary gives a more concise upper bound if ε is small enough with respect to the other parameters.

Corollary 3. *Let $\nu_1 := \big(\frac{\varepsilon}{M_ue}\big)^2\frac{3\mu(1-k)}{(\delta_u+\eta)^3}$. If $\nu_1 \leq 1$, then the entropy of system (2) is upper bounded by: $\frac{nM_x}{\sqrt{k}\ln 2}+$*

$$\frac{1}{\min\Big\{\Big(\frac{\delta_u+\eta}{\mu}\Big)\frac{\nu_1}{3}(1-\frac{\nu_1}{3}), \frac{1}{M_x}\Big\}}\Big(n\log(1+\sqrt{k}) + m\log\lceil\frac{u_{max}}{\delta_u}\rceil\Big). \quad (14)$$

Proof. Since $\nu_1 \leq 1$, $\sqrt[3]{1+\nu_1} > 1 + \frac{\nu_1}{3} - \frac{\nu_1^2}{9}$. Then, $\rho(k,\delta_u)$ is lower bounded by:

$$\Big(\frac{\delta_u+\eta}{\mu}\Big)\Big(-1+1+\frac{\nu_1}{3}-\frac{\nu_1^2}{9}\Big) \geq \Big(\frac{\delta_u+\eta}{\mu}\Big)\frac{\nu_1}{3}(1-\frac{\nu_1}{3}). \quad (15)$$

Thus, if we set T_p to $\min\{\frac{4\nu_1}{3}(1-\frac{\nu_1}{3}), \frac{1}{M_x}\}$, we get $e^{M_xT_p} \leq e$. Moreover, one can easily check that this assignment satisfies the conditions of Lemma 2. If we substitute this value in Proposition 3, we get the corollary. □

Remember that so far we are assuming that $u \in \mathcal{U}(\mu, \eta, u_{max})$, which means it is piecewise-continuous with bounded variation. Now, if we restrict the input signal furthermore to be Lipschitz continuous with Lipschitz constant L_v, then for all $t \geq 0$ and $\tau > 0$, $\|u(t+\tau) - u(t)\| \leq L_v\tau$. This leads to the following corollary.

Corollary 4. *If the input signal u is Lipschitz continuous with Lipschitz constant L_v, the entropy of system (2) has the same upper bounds as in Proposition 4 and Corollary 3 with μ replaced by L_v and η by zero.*

An example: Harrier jet

We study the Harrier "jump jet" model from [1]. The dynamics of the system is given by:

$$\dot{x}_1 = x_2;\ \dot{x}_2 = -g\sin\theta_1 - \frac{c}{m'}x_2 + \frac{u_1}{m'}\cos\theta_1 - \frac{u_2}{m'}\sin\theta_1$$
$$\dot{y}_1 = y_2;\ \dot{y}_2 = g(\cos\theta_1 - 1) - \frac{c}{m'}y_2 + \frac{u_1}{m'}\sin\theta_1 + \frac{u_2}{m'}\cos\theta_1$$
$$\dot{\theta}_1 = \theta_2;\ \dot{\theta}_2 = \frac{r}{J}u_1,$$

where (x_1, y_1, θ_1) are the position and the orientation of the center of mass of the aircraft in the vertical plane, and (x_2, y_2, θ_2) are the corresponding time derivatives. The mass of the aircraft is m', the moment of inertia is J, the gravitational constant is g, and the damping coefficient is c. The Harrier uses maneuvering thrusters for vertical take-off and landing. The inputs u_1 and u_2 are the force vectors generated by the main downward thruster and the maneuvering thrusters.

To compute the upper bound on entropy, we need to find the parameters M_x, M_u, u_{max}, μ, and η for the system. To compute M_x, we compute the Lipschitz constant of f with respect to each of the coordinates in the state vector. Then, we use Proposition 2, to get $M_x = nL'_x + \frac{1}{2}$. To compute the Lipschitz constant, we compute the partial derivative of f with respect to each coordinate and use an upper bound on the infinity norm of each of the resulting vectors. We get $L'_x = g + 2\frac{u_{max}}{m'}$ to be the maximum of these norms, and thus $M_x = 6g + 12\frac{u_{max}}{m'} + \frac{1}{2}$. We get $M_u = 2\sqrt{2}L'_u$ in a similar manner.

Fixing $u_{max} = 50$, $m' = 100$, $g = 9.81$, $r = 5$, and $J = 50$, we get $M_x = 83.36$ and $M_u = 0.2828$. Now, suppose that $\mu = 10$ and $\eta = 20$ and the needed estimation accuracy $\varepsilon = 0.5$. Therefore, if we choose k to be equal to $\frac{1}{2}$ and δ_u to be equal to η (i.e. 20), then $\nu_1 = 9.915 \times 10^{-5}$, $\left(\frac{\delta_u+\eta}{\mu}\right)\frac{\nu_1}{3}(1 - \frac{\nu_1}{3}) = 1.32 \times 10^{-4} \le \frac{1}{M_x} = 0.012 \le 1$. Then, using Corollary 3, we get $h_{\text{est}}(0.5, K) \le 60017$. We get the same upper bound if we instead use Proposition 4. However, if $\mu = 0.1$ and $\eta = 45$, and letting again $\delta_u = \eta$, ν_1 will be equal to 8.7047×10^{-8} and the bound will be equal to 254879. Hence, the bound increased significantly as the the bounds on the variation of the input signal were relaxed. Suppose now that we take the other extreme, where we restrict the allowed size of jumps in the input signal by decreasing η to 0.1 while allowing large continuous variations by increasing μ to 20. In that case, the input signals are almost continuous. ν_1 will be equal to 1586.44 so Corollary 3 can not be used and we should resort to Proposition 4 to get a bound. One can compute $\rho(k, \delta_u)$ to get 0.1067 which is larger than $\frac{1}{M_x} = 0.012$. Hence, the entropy is bounded by 2515 which is much less than the two previous bounds. The effect of the different parameters on the entropy upper bound will be discussed formally in the next section.

4.4 Entropy upper bound discussion

In this section we discuss how the bounds in Proposition 4 and Corollary 3 vary while varying the different system parameters. Remember that μ and η bound the variation of the input signal and u_{max} bounds its norm at any instant in time.

(1) The upper bounds in Proposition 4 and Corollary 3 increase quadratically with η. That is expected as larger jumps in the input signal would lead to a higher uncertainty in the system's state.
(2) As μ increases, the bound in Corollary 3 will increase in the order of $\frac{1}{1-O(\mu)}$ while the bound in Proposition 4 will increase as $O(\mu^{2/3})$.
(3) The bounds in both the proposition and the corollary increase logarithmically in u_{max}. This means that the growth in the uncertainty in the state estimate because of the increase in the bound on the input is at least exponentially slower than the growth caused by its faster variation.
(4) Finally, as ε goes to zero, the upper bound in Proposition 4 grows as $\Omega(\varepsilon^{-2/3})$ and that of Corollary 3 as $\Omega(\varepsilon^{-2})$.

5 SYSTEMS WITH LINEAR INPUTS

In this section, we provide tighter bounds on entropy than that of Proposition 4 for systems where the input affects the dynamics linearly. Formally, we consider dynamical systems of the form:

$$\dot{x}(t) = f(x(t)) + u(t), \tag{16}$$

where the initial state $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$, as before.

We will show in the next section a new IS discrepancy function designed to utilize the linear relation between the input and the state dynamics of the system. Then, in the following section, we will use Algorithm 1 to construct ε-approximating functions for the trajectories of this system using the new IS discrepancy function. After that, we will show that the number of functions that can be constructed by the modified algorithm is the same as that of Lemma 3 in terms of its parameters δ_x, δ_u and T_p. However, larger values of these parameters would suffice to get ε-approximating function. Finally, we will compute the new upper bound and present an example to show the difference between the two bounds.

5.1 Input-to-state discrepancy function construction for systems with linear inputs

In this section, we will show that we can get a tighter upper bound on the distance between two different trajectories than that of (7). Basically, for any two initial states $x_0, x'_0 \in K$, two input signals $u, u' \in \mathcal{U}(\mu, \eta, u_{max})$, and for all $t \in \mathbb{R}_{\ge 0}$,

$$\begin{aligned} &\|\xi_{x_0,u}(t) - \xi_{x'_0,u'}(t)\| \\ &= \|x_0 + \int_0^t \Big(f(\xi_{x_0,u}(s)) + u(s)\Big)ds \\ &\qquad - x'_0 - \int_0^t \Big(f(\xi_{x'_0,u'}(s)) + u'(s)\Big)ds\| \\ &\le \|x_0 - x'_0\| + \int_0^t \|f(\xi_{x_0,u}(s)) - f(\xi_{x'_0,u'}(s))\|ds \\ &\qquad + \int_0^t \|u(s) - u'(s)\|ds \end{aligned}$$

[by triangular inequality]

$$\leq \|x_0 - x_0'\| + \int_0^t L_x \|\xi_{x_0,u}(s) - \xi_{x_0',u'}(s)\| ds + \int_0^t \|u(s) - u'(s)\| ds$$
[by the Lipschitz continuity of f]
$$\leq \Big(\|x_0 - x_0'\| + \int_0^t \|u(s) - u'(s)\| ds\Big) e^{L_x t}, \tag{17}$$

where the last inequality follows from the Bellman-Gronwall inequality. Notice that we have a linear discrepancy function instead of the quadratic one we got in (7). This means that the sensitivity of this system with respect to changes in the input is smaller than that of nonlinear systems in general.

5.2 Approximating set construction

Let us fix $\varepsilon > 0$ for this section. To construct an ε-approximating function for a given trajectory, we use Algorithm 1 again. The following lemma is similar to Lemma 2 as it specifies the conditions that the values of δ_x, δ_u, and T_p should satisfy for the output of Algorithm 1, z, to be an ε-approximating for the system trajectory.

Lemma 4. *Fix a constant $k \in (0,1)$ and the parameters T_p, δ_x, and δ_u, such that:*

(1) $\varepsilon k \geq \delta_x e^{L_x T_p}$, and
(2) $\varepsilon(1-k) \geq T_p(\frac{\mu T_p}{2} + \eta + \delta_u) e^{L_x T_p}$.

Then, for any $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$, for all $i \in [0; \lfloor \frac{T}{T_p} \rfloor]$, and for all $t \in [iT_p, (i+1)T_p)$,

(i) $x_i \in S_i$,
(ii) $\|z_i(t - iT_p) - \xi_{x_i,u_i}(t - iT_p)\| \leq \varepsilon$,

where $u_i(t) := u(iT_p + t)$, the i^{th} piece of the input signal of size T_p.

Proof. Fix $x_0 \in K$ and $u \in \mathcal{U}(\mu, \eta, u_{max})$ and let $t' = t - iT_p$. Then, by (17),

$$\|z_i(t) - \xi_{x_i,u_i}(t)\|$$
$$\leq \Big(\|x_i - q_{x,i}\| + \int_0^{t'} \|u_i(s) - q_{u,i}\| ds\Big) e^{L_x t'}$$
[by (17)]
$$\leq \Big(\|x_i - q_{x,i}\| + \int_0^{t'} \big(\|u_i(s) - u_i(0)\| + \|u_i(0) - q_{u,i}\|\big) ds\Big) e^{L_x t'}$$
[by triangular inequality]
$$\leq \Big(\delta_x + \int_0^{t'} \|u_i(s) - u_i(0)\| ds + t'\delta_u\Big) e^{L_x t'},$$
[since $\|x_i - q_{x,i}\| \leq \delta_x$, $\|u_i(0) - q_{u,i}\| \leq \delta_u$]
$$\leq \Big(\delta_x + \int_0^{t'} (\mu s + \eta) ds + t'\delta_u\Big) e^{L_x t'}$$
[by (1)]
$$\leq \Big(\delta_x + T_p\delta_u + T_p(\frac{\mu T_p}{2} + \eta)\Big) e^{L_x T_p}$$
[since $t' \leq T_p$]
$$\leq k\varepsilon + (1-k)\varepsilon = \varepsilon, \tag{18}$$

where the last inequality follows from the assumption in the Lemma on T_p, δ_x and δ_u. Hence, for all $i \in [0, \lfloor \frac{T}{T_p} \rfloor]$ and $t \in [iT_p, (i+1)T_p]$, $x_i \in S_i$ and $\|\xi_{x_i,u_i}(t) - \xi_{q_{x,i},q_{u,i}}(t)\| \leq \varepsilon$. □

Corollary 5. *Under the same conditions of Lemma 4, the output z of Algorithm 1 is an ε-approximating function of the corresponding trajectory of system (16). Moreover, since we are still using Algorithm 1 to construct the approximating function, we have the same upper bound on entropy of system (16) as in Proposition 3 in terms of the new values k, δ_u and T_p that satisfy the new constraints.*

5.3 Entropy upper bound on systems with linear inputs

It follows from the last corollary in the previous section that we can substitute the upper bounds on the parameters δ_u, δ_x and T_p assumed in Lemma 4 to get the new upper bound. This is shown in the following proposition.

Proposition 5. *Fix $\varepsilon > 0$, $k \in (0,1)$ and $\delta_u \in (0, u_{max}]$ and let*

$$\rho(k, \delta_u) = \Big(\frac{\eta + \delta_u}{\mu}\Big)\Big(-1 + \sqrt{1 + \frac{2\mu(1-k)\varepsilon}{(\eta + \delta_u)^2}}\Big). \tag{19}$$

Then, the entropy of system (16) is upper bounded by:

$$\frac{nM_x}{k \ln 2} + \frac{1}{\min\{\rho(k, \delta_u), 1/L_x\}}\Big(n \log(1+k) + m \log\lceil \frac{u_{max}}{\delta_u} \rceil\Big). \tag{20}$$

For example, k and δ_u can be 1/2 and η, respectively.

Proof. This proof is almost the same as that of that of Proposition 4. Let us assume first that $T_p \leq 1/L_x$, then $e^{L_x T_p}$ is upper bounded by e. In that case, to get a value of T_p that satisfies the condition of Lemma 4, we solve the following polynomial inequality:

$$\frac{\mu {T_p}^2}{2} + T_p(\eta + \delta_u) - (1-k)\varepsilon \leq 0, \tag{21}$$

which has the following roots:

$$\Big(\frac{\eta + \delta_u}{\mu}\Big)\Big(-1 \pm \sqrt{1 + \frac{2\mu(1-k)\varepsilon}{(\eta + \delta_u)^2}}\Big). \tag{22}$$

First, note that the smaller root is negative. Thus, assigning T_p to any value between zero and the larger root, $\rho(k, \delta_u)$ would satisfy the conditions of Lemma 4. Hence, if $\rho(k, \delta_u) \leq 1/L_x$, and we assign T_p to it, we get the first bound in the proposition. If $\rho(k, \delta_u) > 1/L_x$, assigning T_p to $1/L_x$ would still satisfy the conditions of Lemma 4. Hence, we get the second part of the bound. □

As before, we can get a more concise bound if ε is small enough with respect to the other parameters. This is shown in the following corollary.

Corollary 6. *Let $v_2 = \frac{2\mu(1-k)\varepsilon}{(\eta+\delta_u)^2}$. If $v_2 \leq 1$, then the entropy of system (16) is upper bounded by: $\frac{nM_x}{k \ln 2} +$*

$$\frac{1}{\min\Big\{\Big(\frac{\delta_u + \eta}{\mu}\Big)\frac{v_2}{2}(1 - \frac{v_2}{4}), \frac{1}{L_x}\Big\}}\Big(n \log(1+k) + m \log\lceil \frac{u_{max}}{\delta_u} \rceil\Big). \tag{23}$$

Proof. Since $\sqrt{1+\nu_2} \geq 1 + \frac{\nu_2}{2} - \frac{\nu_2^2}{8}$ if $\nu \leq 1$, the larger root is lower bounded by:

$$\Big(\frac{\eta + \delta_u}{\mu}\Big)\Big(-1 + 1 + \frac{\nu_2}{2} - \frac{\nu_2^2}{8}\Big)$$
$$= \Big(\frac{\eta + \delta_u}{\mu}\frac{\nu_2}{2}\Big)\Big(1 - \frac{\nu_2}{4}\Big). \tag{24}$$

Setting T_p to this value in the conditions of Lemma 4 shows that they are satisfied. Moreover, substituting these values instead of $\rho(k, \delta_u)$ in the bound of Proposition 5 results in the bound. □

In the following, we show how to compute the derived upper bound for a standard example in the dynamical systems literature and compare the values of the two upper bounds that we can get for the same example.

A second example: Pendulum

Consider a pendulum system:

$$\dot{x}_1 = x_2;\ \dot{x}_2 = -\frac{Mgl}{I}\sin x_1 + \frac{u}{I},$$

where I is the moment of inertia of the pendulum around the pivot point, u is its input from a DC motor, x_1 is the angular position (with respect to y-axis), x_2 is the angular speed, and l is the length, and M is the mass.

Consider the case when $\frac{Mgl}{I} = 0.98$, $I = 1$, $u_{max} = 2$, $\mu = 0.1$ and $\eta = 1$. Jacobians of f are:

$$J_x = \begin{bmatrix} 0 & 1 \\ -\frac{Mgl}{I}\cos x_1 & 0 \end{bmatrix}\ J_u = \begin{bmatrix} 0 & 0 \\ 0 & 1/I \end{bmatrix}.$$

Hence, $|||J_x|||_\infty = 1$ and thus $M_x = \frac{3}{2}$, $|||J_u||| = \lambda_{max}(J_u^T J_u) = I^2 = 0.96$, and $M_u = 0.96$. We shall compute the entropy bounds for estimation accuracy $\varepsilon = 0.01$.

Hence, if we use the bound of Proposition 4 or Corollary 3, we get $\nu_1 = 2.75 \times 10^{-7}$ and $\left(\frac{\delta_u+\eta}{\mu}\right)\frac{\nu_1}{3}(1 - \frac{\nu_1}{3}) = 1.836 \times 10^{-6} \leq \frac{1}{M_x} = 0.667$, which means $h_{\text{est}}(0.01, K) \leq 1385442$ bps. Since the input linearly affects the dynamics, we can also use Proposition 5, which gives $\nu_2 = 2.5 \times 10^{-4}$ and $\left(\frac{\delta_u+\eta}{\mu}\right)\frac{\nu_2}{2}(1 - \frac{\nu_2}{4}) = 5 \times 10^{-3}$ and hence $h_{\text{est}}(0.01, K) \leq 515$ bps. As we can see from this example, the new bound can be much tighter than that in Proposition 4.

6 CONCLUSION AND FUTURE DIRECTIONS

We presented a notion of topological entropy as a lower bound on the needed bit rate to estimate the state of a nonlinear dynamical system with inputs. We computed an upper bound on entropy and discussed how the different systems parameters, namely μ, η, u_{max} and ε, affect it. We showed that we recover (within a $O(n)$ factor) the upper bound on estimation entropy of autonomous systems in [10] as the bound on the input decreases to zero. We applied these results to compute the bit-rate needed to estimate the states of two example systems. We also showed how the bit-rate estimates can be improved when the inputs enter linearly. In the future, we plan to apply this theory to get bounds on the entropy of switched systems with bounded and average dwell times and to apply it to a network of dynamical systems.

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A APPENDIX: PROOFS OF LEMMAS

This generalization of the mean-value theorem is used in the construction of the local IS discrepancy functions in [8] restricted to time-invariant systems rather than general time variant ones.

Proposition 6. *For any differentiable $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, for any $x, x' \in \mathbb{R}^n$, any $u, u' \in \mathbb{R}^m$:*

$$f(x',u') - f(x,u) = \Big(\int_0^1 J_x(x + (x'-x)s, u')ds\Big)(x'-x) + \Big(\int_0^1 J_u(x, u + (u'-u)\tau)d\tau\Big)(u'-u).$$

Lemma 1. *The function $V(x,x') := \|x - x'\|^2$ is a local IS discrepancy for system (2) over any compact set $\mathcal{X} \subset \mathbb{R}^n$ and interval $[t_0, t_1] \subseteq \mathbb{R}_{\geq 0}$, with*

$$\beta(y, t - t_0) := e^{2a(t-t_0)}y^2 \text{ and } \gamma(y) := b^2 e^{2a(t_1-t_0)}y^2,$$

where $t \in [t_0, t_1]$,

$$a := \sup_{\substack{t\in[t_0,t_1] \\ u\in\mathcal{U}(\mu,\eta,u_{max}), x\in\mathcal{X}}} \lambda_{max}\Big(\frac{J_x + J_x^T}{2}\Big) + \frac{1}{2} \text{ and}$$
$$b := \sup_{\substack{t\in[t_0,t_1] \\ u\in\mathcal{U}(\mu,\eta,u_{max}), x\in\mathcal{X}}} |||J_u|||. \tag{6}$$

Proof. Let x and $x' \in K$, and u and $u' \in \mathcal{U}$. Define $y(t) = \xi_{x',u'}(t) - \xi_{x,u}(t)$ and $v(t) = u'(t) - u(t)$. For a $t \in \mathbb{R}_{\geq 0}$, using proposition (6), we have

$$\begin{aligned}\dot{y}(t) &= f(\xi_{x',u'}(t), u'(t)) - f(\xi_{x,u}(t), u(t)),\\ &= \Big(\int_0^1 J_x(\xi_{x,u}(t) + sy(t), u'(t))ds\Big)y(t)\\ &\quad + \Big(\int_0^1 J_u(\xi_{x,u}(t), u(t) + v(t)\tau)d\tau\Big)v(t).\end{aligned} \tag{25}$$

We write $J_x(\xi_{x,u}(t) + sy(t), u'(t))$ as $J_x(t,s)$ or simply J_x when the dependence on t and s is clear from context. Similarly, $J_u(\xi_{x,u}(t), u(t) + v(t)\tau)$ is written as $J_u(t,\tau)$ or J_u. Then, differentiating $\|y(t)\|^2$ with respect to t leads to:

$$\begin{aligned}\frac{d}{dt}\|y(t)\|^2 &= \frac{d}{dt}(y(t)^T y(t)) = \dot{y}(t)^T y(t) + y(t)^T \dot{y}(t)\\ &= y(t)^T\Big(\int_0^1 (J_x^T + J_x)ds\Big)y(t) + v(t)^T\Big(\int_0^1 J_u^T d\tau\Big)y(t)\\ &\quad + y(t)^T\Big(\int_0^1 J_u d\tau\Big)v(t)\\ &\qquad\qquad [\text{substituting } \dot{y}(t) \text{ with (25)}]\\ &\leq y(t)^T\Big(\int_0^1 (J_x^T + J_x)ds\Big)y(t) + y(t)^T y(t)\\ &\quad + \Big(\Big(\int_0^1 J_u d\tau\Big)v(t)\Big)^T\Big(\Big(\int_0^1 J_u d\tau\Big)v(t)\Big),\end{aligned} \tag{26}$$

where the inequality follows from the fact that for all $w, z \in \mathbb{R}^n$, $w^T z + z^T w \leq w^T w + z^T z$, since $0 \leq (z-w)^T(z-w) = z^T z - w^T z - z^T w + w^T w$. Let $\lambda_J(\mathcal{X}) = \sup_{x\in\mathcal{X}} \lambda_{max}(\frac{J_x + J_x^T}{2})$ be the upper bound of the eigenvalues of the symmetric part of J_x over $\mathcal{X}$, so $J_x + J_x^T \leq 2\lambda_J(K)I$. Thus, (26) becomes:

$$\begin{aligned}\frac{d}{dt}\|y(t)\|^2 &\leq (2\lambda_J(\mathcal{X}) + 1)\|y(t)\|^2 + \|\Big(\int_0^1 J_u d\tau\Big)v(t)\|^2\\ &\leq 2a\|y(t)\|^2 + (b\|v(t)\|)^2,\end{aligned}$$

for $t \in [t_0, t_1]$. Finally, by integrating both sides of the above inequality from t_0 to t and using Bellman-Gronwall inequality, we get: $\|y(t)\|^2 \leq e^{2a(t_1-t_0)}\Big(\|y(0)\|^2 + \int_{t_0}^t (b\|v(\tau)\|)^2 d\tau\Big)$. □

Proposition 2. *For any time interval $[t_0, t_1] \subset \mathbb{R}_{\geq 0}$ and compact set $\mathcal{X} \subset \mathbb{R}^n$, $a \leq nL'_x + \frac{1}{2}$ and $b \leq m\sqrt{m}L'_u$, where L'_x and L'_u are the Lipschitz constants of f with respect to each coordinate of the state and the input respectively.*

Proof. First, J_u and J_x exist since f is differentiable in both arguments. Second, note that $|||J_u||| \leq \sqrt{m}|||J_u|||_\infty$, where $|||J_u|||_\infty = \max_{i\in[n]} \sum_{j=1}^m |(J_x)_{i,j}|$, and $(J_u)_{i,j}$ is the entry in the i^{th} row and j^{th} column of J_u. Moreover, since for all $i \in [n], j \in [m]$, $|(J_u)_{i,j}| \leq L'_u$, by Lipschitz continuity of f with respect to u, then $|||J_u|||_\infty \leq mL'_u$. Hence, $|||J_u||| \leq m\sqrt{m}L'_u$. Similarly, one can prove that $|||J_x|||_\infty \leq nL'_x$, since the number of columns is n instead of m. Therefore,

$$a \leq |||\frac{J_x + J_x^T}{2}|||_\infty + \frac{1}{2} \leq \frac{|||J_x|||_\infty + |||J_x^T|||_\infty}{2} + \frac{1}{2} \leq nL'_x + \frac{1}{2}, \text{ and,}$$
$$b \leq m\sqrt{m}L'_u.$$

□