State Estimation of Dynamical Systems with Unknown Inputs: Entropy and Bit Rates

Hussein Sibai and Sayan Mitra

HSCC'18 – April 13th, 2018

University of Illinois at Urbana-Champaign

Coordinated Science Laboratory

Motivation: Harrier Jump Jet[1]

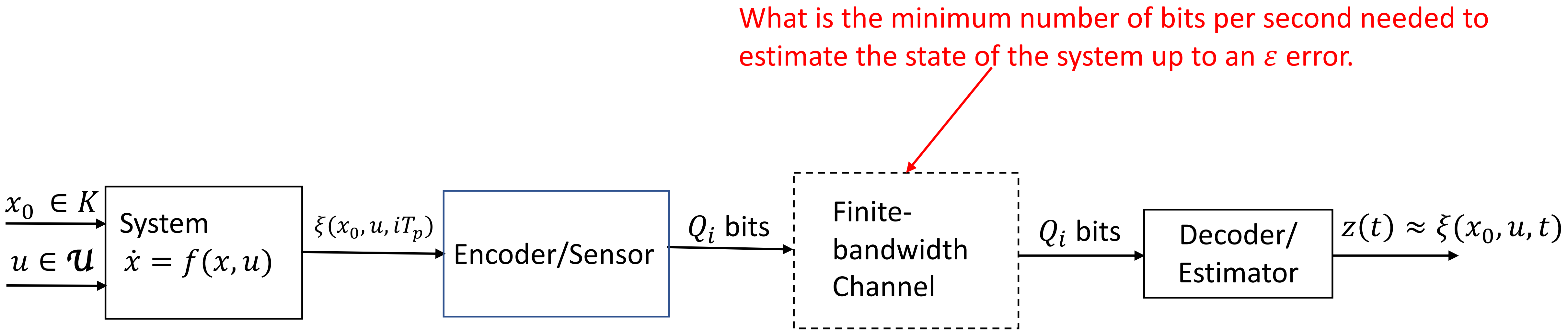
$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -g\sin\theta_1 - \frac{c}{m}x_2 + \frac{u_1}{m}\cos\theta_1 - \frac{u_2}{m}\sin\theta_1,$$
$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = g(\cos\theta_1 - 1) - \frac{c}{m}y_2 + \frac{u_1}{m}\sin\theta_1 + \frac{u_2}{m}\cos\theta_1,$$
$$\dot{\theta}_1 = \theta_2,$$
$$\dot{\theta}_2 = \frac{r}{J}u_1.$$

State variables: $x_1, x_2, y_1, y_2, \theta_1, \theta_2$
Input variables: u_1, u_2

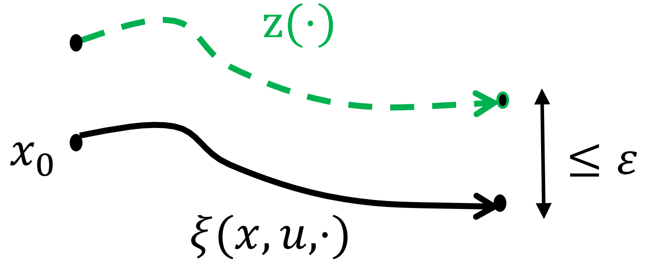
What is the bit rate between the jet and the command center needed to estimate the state of the jet up to an ϵ error?

(1): [Karl Johan Astrom and Richard M. Murray. 2008. Feedback Systems: An Introduction for Scientists and Engineers. Princeton University Press, Princeton, NJ, USA]

Problem Setup



- f globally Lipschitz in both arguments with constants L_x and L_u
- K a compact set in $\mathbb{R}^n$
- $\mathcal{U}$ a set of input signals (will be defined next) in $\mathbb{R}^m$



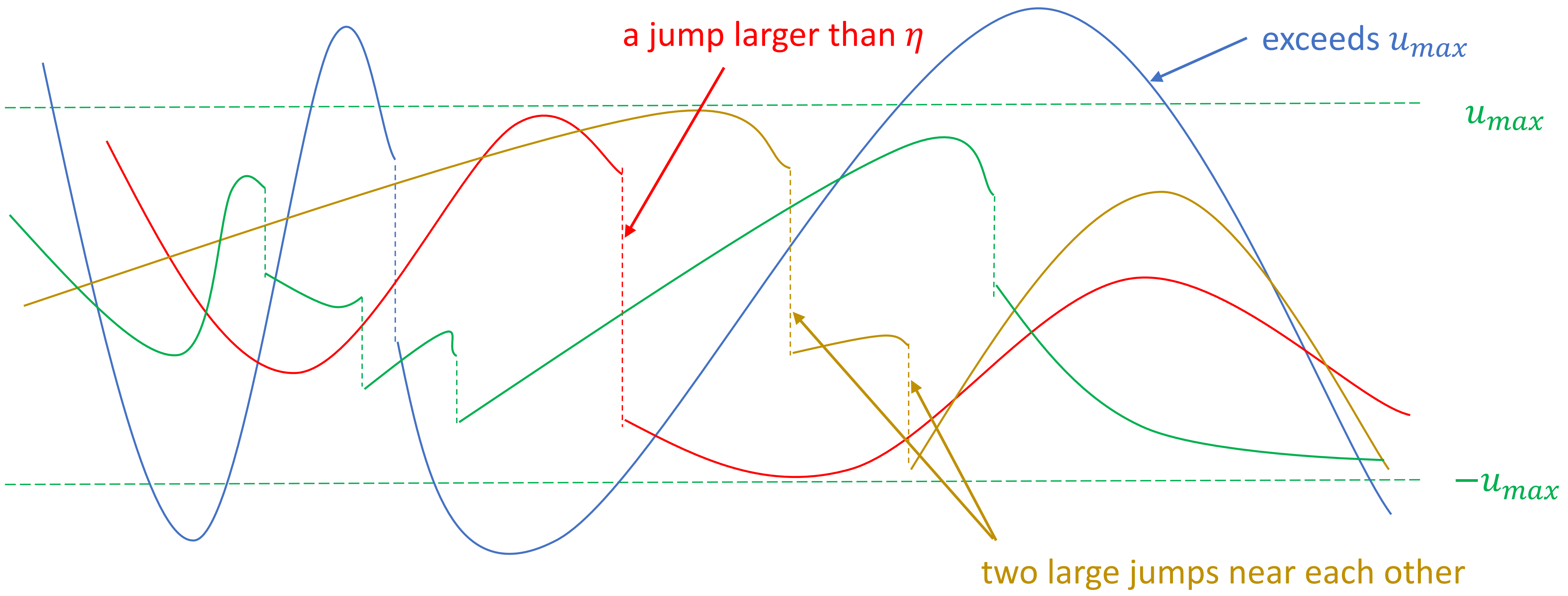
Related Work

- Entropy and Minimal Data Rates for State Estimation and Model Detection *[Liberzon and Mitra, HSCC'16, CDC'16, TAC'17]*
  - Autonomous dynamical systems (no inputs), exponential convergence of error
- Optimal Data Rate for State Estimation of Switched Nonlinear Systems *[Our paper in HSCC'17]*
  - Switched systems, finite number of modes, combination between constant and exponentially converging error

[A. V. Savkin. Automatica'06, F. Colonius. SIAM'12, M. Rungger and M. Zamani. HSCC'17…]

Space of Input Signals: $\mathcal{U}(\eta, \mu, u_{max})$

Given η, μ and $u_{max} \geq 0$, if $u: [0, \infty) \to \mathbb{R}^m$ belongs to $\mathcal{U}(\eta, \mu, u_{max})$, then $\forall t, \tau \geq 0, ||u(t) - u(t + \tau)|| \leq \mu\tau + \eta$ and $|u(t)| \leq u_{max}$.

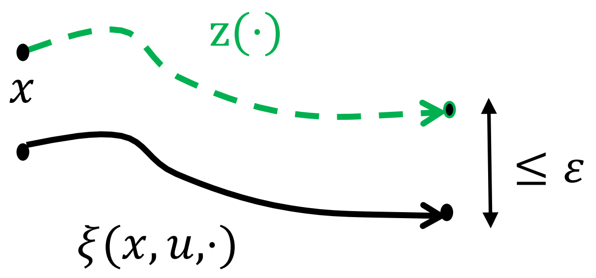


Entropy Definition

A function of the system dynamics and the allowed estimation error that lower bounds the needed bit rate of the channel

Approximating Functions

$z: \mathbb{R}^+ \to \mathbb{R}^n$ is an (ε, T)-approximating function for $\xi(x_0, u, t)$ if:
$\|\xi(x_0, u, t) - z(t)\| \leq \varepsilon$ for all $t \in [0, T]$.



Approximating Sets and Entropy

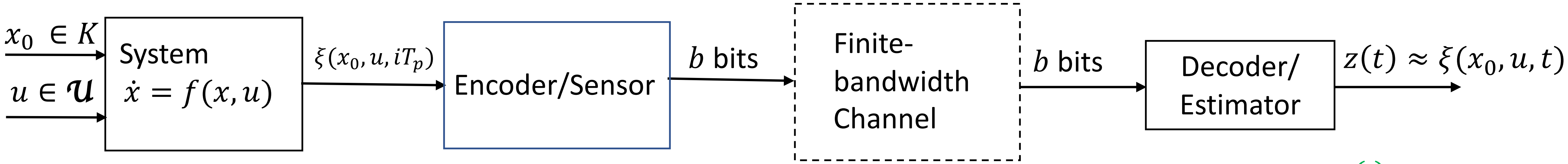
$\hat{Z} = \{\hat{z}_1, \hat{z}_2, \ldots \hat{z}_M\}$ is a (T, ε, K)-approximating set if: $\forall x_0 \in K$ and $u \in \mathcal{U}(\eta, \mu, u_{max})$, there exists $\hat{z}_i \in \hat{Z}$ that is (ε, T)-approximating for $\xi(x_0, u, t)$ over $[0, T]$.

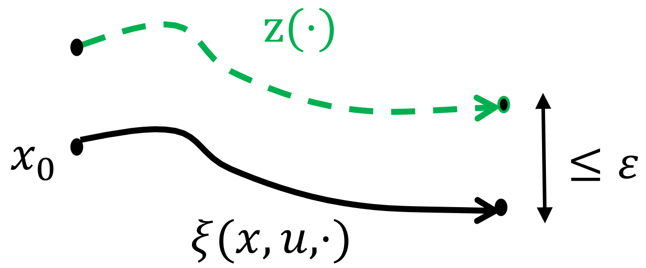
$s_{est}(T, \varepsilon, K)$ is the minimum cardinality of such approximating set.

Entropy:

$$h_{est}(\varepsilon, K) := \limsup_{T \to \infty} \frac{1}{\mathrm{T}} \log \mathrm{s}_{\mathrm{est}}(\mathrm{T}, \varepsilon, \mathrm{K}).$$

State Estimation Algorithms with Fixed Bit Rates





Average bit rate: $b_r(\varepsilon, K) = \frac{b}{T_p}$.

Impossibility of Estimating Below Entropy Rates

First result:

Theorem 1: There is no algorithm with fixed bit rate that, for any $x_0 \in K, u \in \mathcal{U}(\eta, \mu, u_{max})$ and $T \geq 0$, constructs an (ε, T)-approximating function for the trajectory $\xi(x_0, u, \cdot)$ while achieving a bit rate less than the entropy of the system.

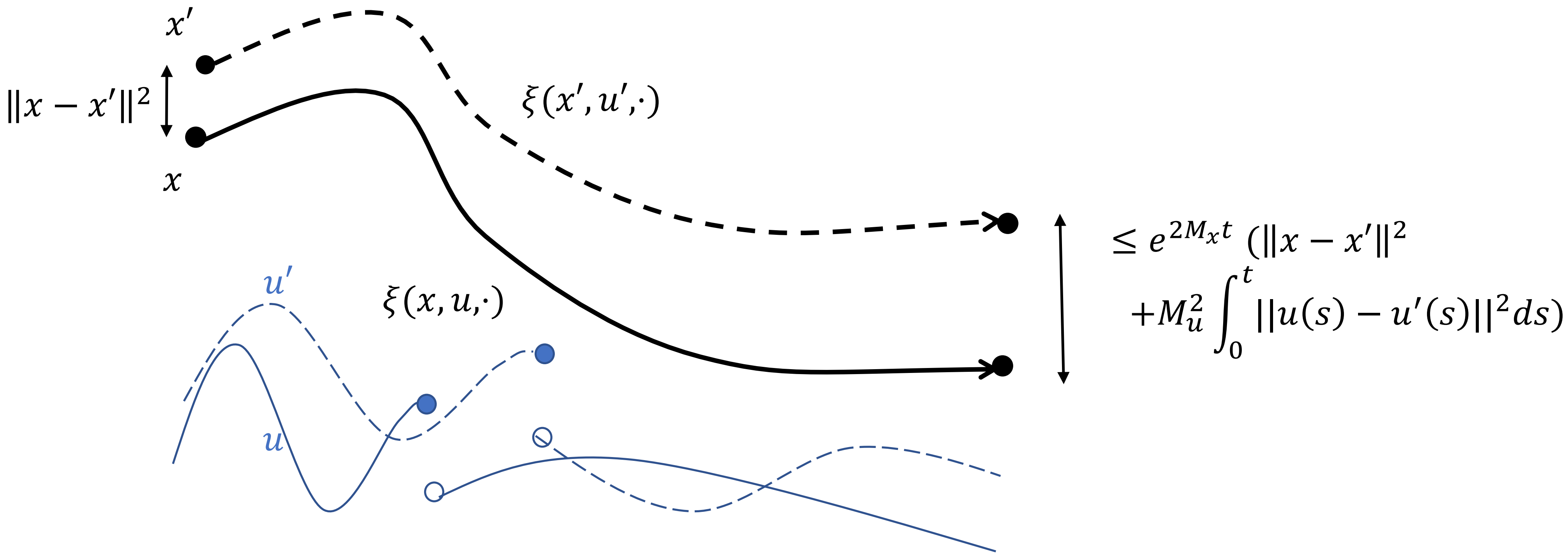
Proof: Existence of such algorithm implies the existence of an (ε, T, K)-approximating set of cardinality smaller than s_{est}, contradiction.

Upper Bound on Entropy of Nonlinear Systems

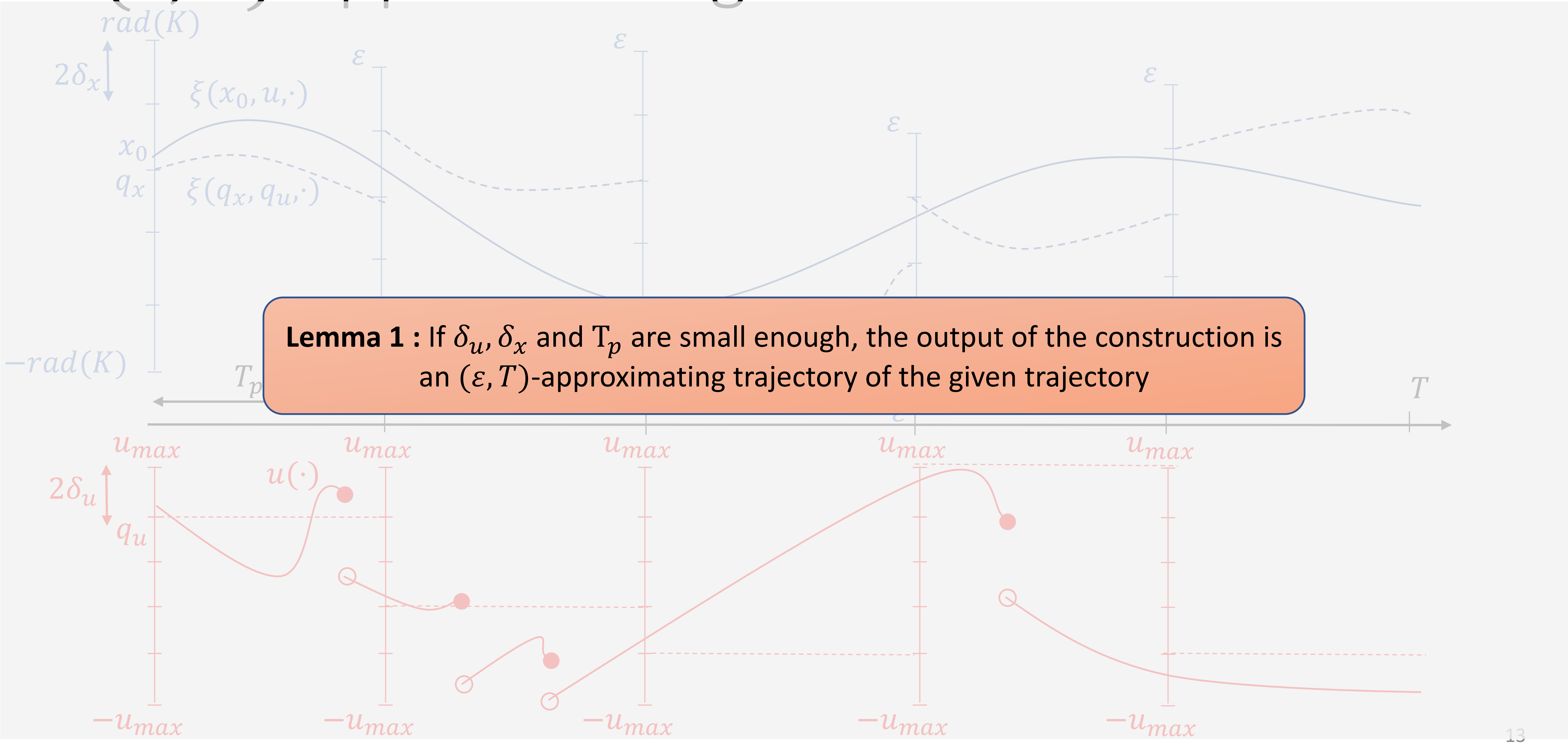
Computing entropy is hard. Computing bounds is easier.

Distance between trajectories after t seconds

$$M_x = nL_x + \frac{1}{2}; M_u = m\sqrt{m}L_u$$



(ε, T)-approximating function construction



Lemma 1 : If δ_u, δ_x and T_p are small enough, the output of the construction is an (ε, T)-approximating trajectory of the given trajectory

(K, ε, T) −approximating set construction

- As x_0 and u vary, what is the number of functions that can be constructed by the algorithm in the previous slide?
- At the first step, the construction chooses:
  - One from $\left(\frac{diam(K)}{2\delta_x}\right)^n$ possible quantization points
  - One from $\left(\frac{u_{max}}{\delta_u}\right)^m$ possible quantization points
- At each time step of size T_p, the construction chooses:
  - One from $\left(\frac{\epsilon}{\delta_x}\right)^n$ possible quantization points in the state space
  - One from $\left(\frac{u_{max}}{\delta_u}\right)^m$ possible quantization points in the input space
- There are $\left\lceil\frac{T}{T_p}\right\rceil$ time steps

(K, ε, T) –approximating set construction

The number of functions that can be constructed by the procedure is upper bounded by:

$$\left(\frac{diam(K)}{2\delta_x}\right)^n \left(\frac{u_{max}}{\delta_u}\right)^m \left(\left(\frac{\epsilon}{\delta_x}\right)^n \left(\frac{u_{max}}{\delta_u}\right)^m\right)^{\left\lfloor\frac{T}{T_p}\right\rfloor}$$

Intermediate upper bound on entropy

Substituting the bound on the cardinality of the approximating set in the entropy definition while fixing $\delta_x = \frac{\varepsilon}{2} e^{-M_x T_p}$, leads to:

$$h_{est}(\varepsilon, K) \leq \frac{2nM_x}{\ln 2} + \frac{1}{T_p}\left(\text{n} \log 2 + m \log \frac{u_{max}}{\delta_u}\right)$$

As $u_{max} \to 0, h_{est}(\varepsilon, K) \leq \frac{nM_x}{\ln 2}$, similar the earlier bound $\frac{nL_x}{\ln 2}$ by Liberzon and Mitra in HSCC'16 (remember $M_x \leq nL_x + \frac{1}{2}$)

Entropy upper bound

Fix an $\varepsilon > 0, h_{est}(\varepsilon, K)$ is upper bounded by:

$$\frac{2nM_x}{\ln 2} + \frac{1}{\min\left\{\rho(\mu,\eta,\varepsilon), \frac{1}{M_x}\right\}}\left(\mathrm{n}\log 2 + m\log\frac{u_{max}}{\eta}\right),$$

where $\rho(\mu, \eta, \varepsilon) = \frac{2\eta}{\mu}\left(-1 + \left(1 + \left(\frac{\varepsilon}{M_u e}\right)^2 \frac{9\mu}{32\eta^3}\right)^{\frac{1}{3}}\right)$ and $M_x = nL_x + \frac{1}{2}$

Remember:

u_{max} is the maximum $||\cdot||_\infty$ of the input signal,

$||u(t) - u(t+\tau)||_\infty \leq \mu\tau + \eta$ and

n and m are the state and input dimensions.

Upper bound discussion

- It increases *quadratically* with η
- It increases as $\frac{1}{1 - O(\mu)}$ with μ
- It increases *logarithmically* with u_{max}
- It increases as $\Omega(\varepsilon^{-\frac{2}{3}})$ as ε goes to zero

Back to the Harrier Jump Jet Example

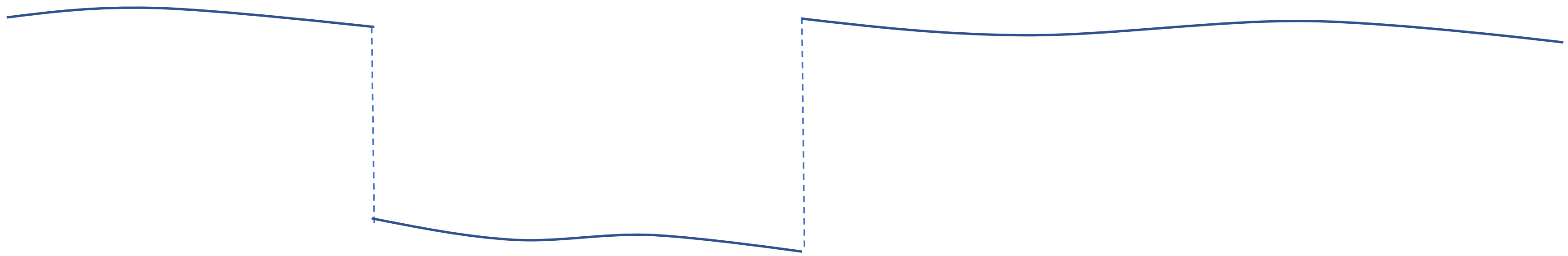
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g\sin\theta_1 - \frac{c}{m}x_2 + \frac{u_1}{m}\cos\theta_1 - \frac{u_2}{m}\sin\theta_1$$
$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = g(\cos\theta_1 - 1) - \frac{c}{m}y_2 + \frac{u_1}{m}\sin\theta_1 + \frac{u_2}{m}\cos\theta_1$$
$$\dot{\theta}_1 = \theta_2$$
$$\dot{\theta}_2 = \frac{r}{J}u_1$$

State variables: $x_1, x_2, y_1, y_2, \theta_1, \theta_2$
Input variables: u_1, u_2

What is the bit rate between the jet and the command center needed to estimate the state of the jet up to an ϵ error? $h_{est}(\varepsilon, K) \leq 61$ Kbps (when $\varepsilon = 0.5, \mu = 10, \eta = 20$)

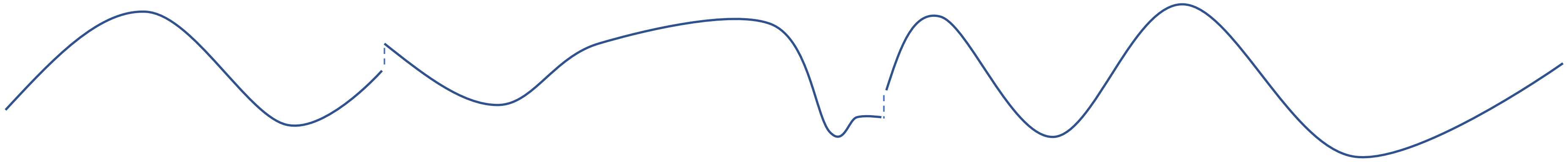
Bound variation as μ and η change

When only small variation of input with large jumps is allowed:



For $\mu = 0.1$ and $\eta = 45$, $h_{est}(\varepsilon, K) \leq 255$ Kbps

When large variation of input with only small jumps is allowed:



For $\mu = 20$ and $\eta = 0.1$, $h_{est}(\varepsilon, K) \leq 2.6$ Kbps

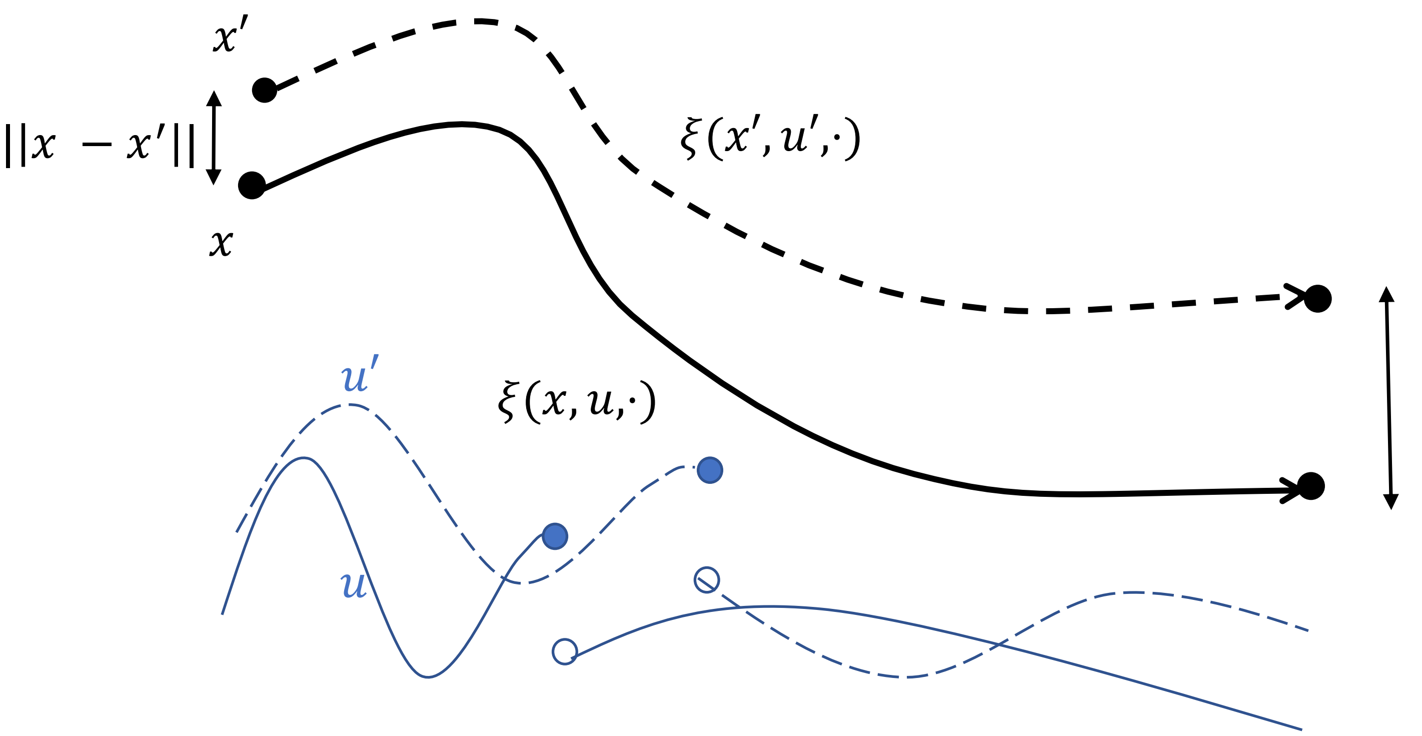
Upper Bound on Entropy of Systems with Linear Inputs

Systems with Linear Inputs

$$\dot{x} = f(x) + u,$$

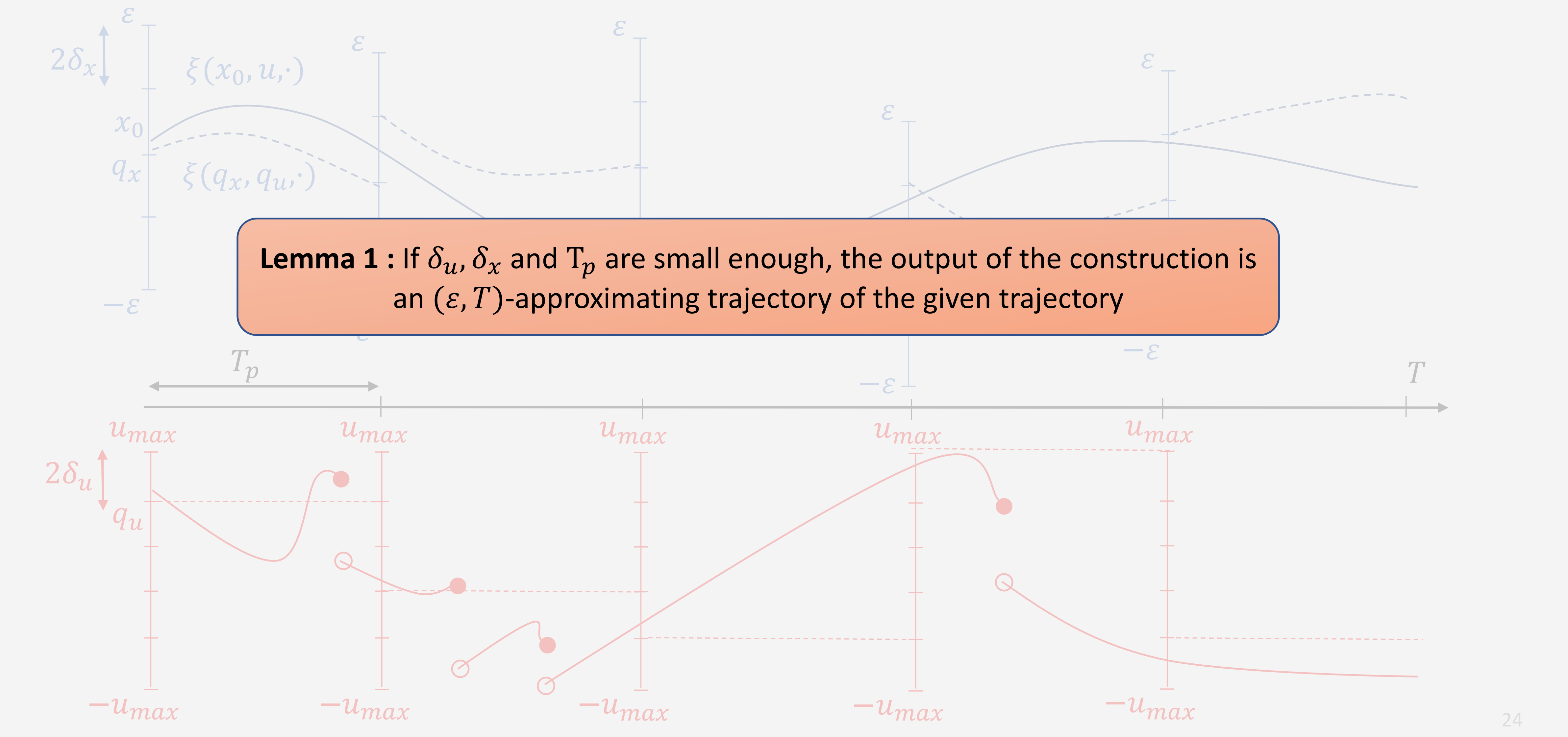
where $x_0 \in K$, a compact set in $\mathbb{R}^n$ and $u \in \mathcal{U}(\eta, \mu, u_{max})$.

Distance between trajectories after t seconds



$$\leq e^{L_x t}\left(\begin{array}{c}||x - x'|| \\ + \int_0^t ||u(s) - u'(s)||ds\end{array}\right)$$

Use the same (ε, T)-approximating function construction



Lemma 1 : If δ_u, δ_x and T_p are small enough, the output of the construction is an (ε, T)-approximating trajectory of the given trajectory

Entropy upper bound

Fix an $\varepsilon > 0, h_{est}(\varepsilon, K)$ is upper bounded by:

$$\frac{2nM_x}{\ln 2} + \frac{1}{\min\left\{\rho(\mu,\eta,\varepsilon),\frac{1}{L_x}\right\}}\left(\mathrm{n}\log 2 + m\log\frac{u_{max}}{\eta}\right),$$

where $\rho(\mu, \eta, \varepsilon) = \frac{2\eta}{\mu}\left(-1 + \sqrt{\left(1 + \frac{\mu\varepsilon}{4\eta}\right)}\right).$

Pendulum example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-Mgl}{I}\sin x_1 + \frac{u}{I}$$

where $Mgl = 0.98, I = 1, u_{max} = 2, \mu = 0.1, \eta = 1$ and $\varepsilon = 0.5$.

Using the upper bound on entropy:

- For general nonlinear systems: $h_{est}(\varepsilon, K) \leq 1386$ Kbps,
- For systems with linear inputs: $h_{est}(\varepsilon, K) \leq 0.6$ Kbps.

The latter bound can be much tighter than the former.

Ongoing Work

Relating the bounds in this paper with previous bounds on switched systems

Thank you