ECE/CS 541: COMPUTER SYSTEM ANALYSIS

Homework #1 – Due Tuesday, September 1, Beginning of Class

August 25, 2008

The intent of this homework is to review material you should have learned in your probability and statistics course (e.g., ECE 413) and explore new material regarding probability theory and stochastic processes we have discussed in class. They vary in difficulty from those you should be able to solve in a couple of minutes, to those that require more thought. If you have trouble with these problems, you should review material from your probability course, or come and talk to me during office hours.

1. Consider the following program segment:

   if B then
       repeat S_1 until B_1
   else
       repeat S_2 until B_2

   Assume that $P(B = true) = p$, $P(B_1 = true) = 5/8$, and $P(B_2 = true) = 3/8$. Exactly one statement is common to statement groups S_1 and S_2: write ('good day'). After many repeated executions of the program segment, it has been estimated that the probability of printing exactly three 'good day' messages is 15/128. Derive the value of p.

2. In manufacturing a certain component, two types of defects are likely to occur with respective probabilities 0.05 and 0.1. Assuming that the defects are independent of one another, what is the probability that a randomly chosen component:

   (a) Does not have **both** defects?
   (b) Is defective?
   (c) Has only one kind of defect, given that it is found to be defective?

3. A mischievous student wants to break into a computer file, which is password-protected. Assume that there are n equally likely passwords, and that the student chooses passwords independently and at random and tries them. Let N_n be the number of trials required to break into the file. Determine the probability density function of N_n (a) if unsuccessful passwords are not eliminated from further selections, and (b) if they are.

4. Assume that the number of messages input to a communication channel in an interval of duration t seconds is Poisson distributed with parameter $0.3t$. Compute the probabilities of the following events:
   (a) Exactly two messages will arrive during a five-second interval,
   (b) At most eight messages arrive in a period of ten seconds,
   (c) The number of message arrivals in an interval of duration twenty seconds is between eight and twelve.

5. Consider a communication channel, where the probability of error-free transmission of a packet has some fixed value p. Suppose further that if a packet is erroneous when received, a retransmission is initiated. This is repeated until an error-free transmission occurs. Moreover, the outcome of each transmission is independent of previous transmissions. To put this procedure in a more formal set-

ting, let X be a random variable, where X = the number of retransmissions required for error-free reception. (If the initial transmission is error-free, $X = 0$.)

(a) Construct a probability space (Ω, ε, P) which suffices to serve as the underlying space for X. (Regarding Ω, you can get by with a sample space that's countably infinite.) With regard to the measure P, for any singleton event $\{\omega\}$ ($\omega \in \Omega$), you should be able to give a formula that expresses the probability $P(\{\omega\})$ in terms of ω. (You need not formulate P for larger events; their values follow from axioms and theorems for a probability space.)

(b) Since X is a discrete-valued random variable, its probabilistic nature can be captured by its probability distribution (PD), i.e., the probabilities $\{P[X = 0], P[X = 1], ...\}$. Give a general formula which, for $k \in \{0, 1, ...\}$, expresses the probability $P[X = k]$. (You may already have computed this as part of your answer to part (a).)

(c) Formulate the expected value $E[X]$ and the variance $\mathrm{Var}[X]$ of this random variable.

6. Let $X = \{X_t \mid t \in T\}$ be an independent stochastic process, let n be a positive integer, let $(t_1, t_2, ... t_n)$ be a sequence of distinct times ($t_i \in T$) and consider the random variable $Y : \Omega \to \Re$ (Ω is the sample space for variables X_t) where

$$Y(\omega) = \text{minimum}\{X_{t_1}(\omega), X_{t_2}(\omega), ..., X_{t_n}(\omega)\}$$

Express the cumulative distribution function (cdf) of Y in terms of
$X_{t_1}, X_{t_2}, X_{t_3}, ... X_{t_n}$.

7. Suppose the variables X_t of the process in the previous problem are identically distributed with an exponential distribution having parameter λ ($\lambda > 0$), i.e., for all t in T,

$$F_{x_t}(x) = 1 - e^{-\lambda x}$$

Determine the cumulative distribution function (cdf) and probability density funtion (pdf) of the random variable Y defined in the previous problem.

8. Let X be the stochastic process considered in Problem 7 and suppose further X is discrete-time with $T = \{0,1,2,...\}$. Consider now the "random sum"

$$S_N = \sum_{t=0}^{N-1} X_t$$

where N is a geometrically distributed random variable, i.e., for $k \in \{1, 2, ...\}$

$$P[N = k] = r(1 - r)^{k-1}$$

with r some real number, $0 < r < 1$. Prove that S_N is exponentially distributed.